ON α-ROUGHLY WEIGHTED GAMES

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ABSTRACT. Gvozdeva, Hemaspaandra, and Slinko (2011) have introduced three hierarchies for simple games in order to measure the distance of a given simple game to the class of (roughly) weighted voting games. Their third class $\mathcal{C}_\alpha$ consists of all simple games permitting a weighted representation such that each winning coalition has a weight of at least 1 and each losing coalition a weight of at most α. For a given game the minimal possible value of α is called its critical threshold value. We continue the work on the critical threshold value, initiated by Gvozdeva et al., and contribute some new results on the possible values for a given number of voters as well as some general bounds for restricted subclasses of games. A strong relation beween this concept and the cost of stability, i.e. the minimum amount of external payment to ensure stability in a coalitional game, is uncovered.

1. INTRODUCTION

For a given set $N = \{1, \ldots, n\}$ of n voters a simple game is a function $\chi : 2^N \to \{0, 1\}$ which is monotone, i.e. $\chi(S) \leq \chi(T)$ for all $S \subseteq T \subseteq N$, and fulfills $\chi(\emptyset) = 0$, $\chi(N) = 1$. Here 2^N denotes the set of all subsets of N. Those subsets are also called coalitions and N is called the grand coalition. By representing the subsets of N by their characteristic vectors in $\{0, 1\}^n$ we can also speak of a (monotone) Boolean function. If $\chi(S) = 1$ then S is called a winning coalition and otherwise a losing coalition. An important subclass is the class of weighted voting games for which there are weights w_i for $i \in N$ and a quota $q > 0$ such that the condition $\sum_{i \in S} w_i \geq q$ implies coalition S is winning and the condition $\sum_{i \in S} w_i < q$ implies coalition S is losing. One attempt to generalize weighted voting games was the introduction of roughly weighted games, where coalitions S with $\sum_{i \in S} w_i = q$ can be either winning or losing independently from each other. [1] As some games being important both for theory and practice are not even roughly weighted, [Gvozdeva et al., 2012] have introduced three hierarchies for simple games to *measure* the *distance* of a given simple game to the class of (roughly) weighted voting games. In this paper we want to study their third class $\mathcal{C}_\alpha$, where the tie-breaking point q is extended to the interval $[1, \alpha]$ for an $\alpha \in \mathbb{R}_{\geq 1}$. Given a game χ, the smallest possible value for α is called the critical threshold-value $\mu(\chi)$ of χ, see the beginning of Section 2. Let $c_{\mathcal{S}}(n)$ denote the largest critical threshold-value within the class of simple games $\chi \in \mathcal{S}_n$ on n voters. By $Spec_{\mathcal{S}}(n) := \{\mu(\chi) \mid \chi \in \mathcal{S}_n\}$ we denote the set of possible critical threshold values.

During the program of classification of simple games, see e.g. [von Neumann and Morgenstern, 2007], several subclasses have been proposed and analyzed. Although weighted voting games are one of the most studied and most simple forms of simple games, they have the shortcomming of not covering all games. The classes $\mathcal{C}_\alpha$ resolve this by introducing a parameter α, so that by varying α the classes of games can be made as large as possible. The critical threshold value in some sense measures the complexity of a given game. Another such measure is the dimension of a simple game, see e.g. [Taylor and Zwicker, 1993]. Here we observe that there is no direct relation between these two concepts, i.e. simple games with dimension 1 have a critical threshold value of 1, but simple games with dimension larger than 1 can have arbitrarily large critical threshold values.

Also graphs have been proposed as a suitable representational language for coalitional games. There are a lot of different graph-based games like e.g. shortest path games, connectivity games, minimum cost spanning tree games, and network flow games. The players of a network flow game are the edges in an edge weighted graph, see [Granot and Granot, 1992] and [Kalai and Zemel, 1982]. For so called threshold network flow games, see e.g. [Bachrach, 2011], a coalition of edges is winning if and only if those edges allow a flow from a given source to a sink which meets or exceeds a given quota or threshold. Here the same phenomenon as for weighted voting games arises, i.e. those graph based *weighted* games are not fully expressive, but general network flow games are (within

[1] Some authors, e.g. [Gvozdeva et al., 2012], allow $q = 0$, which makes sense in other contexts like circuits or Boolean algebra. Later on, we want to rescale the quota q to one, so that we forbid a quota of zero by definition. Another unpleasant consequence of allowing $q = 0$ would be that each simple game on n voters is contained in a roughly weighted game on $n + 1$ voters, i.e., we can add to each given simple game a voter who forms a winning coalition on its own to obtain a roughly weighted game.

the class of stable games). Similarly, one can define a hierarchy by requesting a flow of at least 1 for each winning coalition and a flow of at least α for each losing coalition.

The concept of the cost of stability was introduced in [Bachrach et al., 2009]. It asks for the minimum amount of external payment given to the members of a coaltion to ensure stability in a coalition game, i.e., to guarantee a non-empty core. It will turn out that the cost of stability is closely related to the notion of α-roughly weightedness. For network flow games some results on the cost of stability can be found in [Resnick et al., 2009].

Another line of research, which is related with our considerations, looks at the approximability of Boolean functions by linear threshold functions, see [Diakonikolas and Servedio, 2012].

In [Gvozdeva et al., 2012] the authors have proven the bounds $\frac{1}{2} \cdot \left\lfloor \frac{n}{2} \right\rfloor \leq c_{\mathcal{S}}(n) \leq \frac{n-2}{2}$ and determined the spectrum for $n \leq 6$. For odd numbers of voters we slightly improve the lower bound to $c_{\mathcal{S}}(n) \geq \left\lfloor \frac{n^2}{4} \right\rfloor /n$, which is conjectured to be tight. As upper bound we prove $c_{\mathcal{S}}(n) \leq \frac{n}{3}$. In order to determine the exact values of $c_{\mathcal{S}}(n)$ for small numbers of voters we provide an integer linear programming formulation. This approach is capable to treat cases where exhaustive enumeration is computationally infeasible due to the rapidly increasing number of voting structures. Admittedly, this newly introduced technique, which might be applicable in several other contexts in algorithmic game theory too, is still limited to a rather small number of voters.

From known results on the spectrum of the determinants of binary $n \times n$-matrices we are able to conclude some information on the spectrum of the possible critical threshold values.

The same set of problems can also be studied for subclasses of simple games and we do so for complete simple games, denoted here by $\mathcal{C}$. Here we conjecture that the maximum critical threshold value $c_{\mathcal{C}}(n)$ of a complete simple game on n voters is bounded by a constant multiplied by $\sqrt{n}$ on both sides. A proof could be obtained for the lower bound, and, for some special subclasses of complete simple games, also for the upper bound. In general, we can show that $c_{\mathcal{C}}(n)$ grows slower than any linear function reflecting the valuation that complete simple games are somewhat *nearer* to weighted voting games than general simple games.

The remaining part of this paper is organized as follows: After this introduction we present the basic definitions and results on linear programs determining the critical threshold value of a simple game or a complete simple game in Section 2. In Section 3 we provide certificates for the critical threshold value. General lower and upper bounds on the maximum possible critical threshold values $c_{\mathcal{S}}(n)$ and $c_{\mathcal{C}}(n)$ of simple games and complete simple games are the topic of Section 4. In Section 5 we provide an integer linear programming formulation to determine the exact value $c_{\mathcal{S}}(n)$ and $c_{\mathcal{C}}(n)$. To this end we utilize the dual of the linear program determining the critical threshold value. In Section 6 we give some restrictions on the set of possible critical threshold values and tighten the findings of [Gvozdeva et al., 2012]. We end with a conclusion in Section 7.

2. PRELIMINARIES

In this paper we want to study different classes of *voting structures*. As abbreviation for the most general class we use the notation $\mathcal{B}_n$ for the set of Boolean functions $f : 2^N \to \{0, 1\}$ with $f(\emptyset) = 0$ on n variables[2] As a shortcut for the sum of weights $\sum_{i \in S} w_i$ of a coalition $S \subseteq N$ we will use $w(S)$ in the following.

In this section we state the preliminaries, i.e., we define the mentioned classes of voting structures and provide tailored characterizations of the criticial threshold value within these classes. As a first result we determine the largest possible critical threshold value for Boolean functions in Lemma 1. Since it is closely related, we briefly introduce the concept of the cost of stability for binary voting structures.

Definition 1. *A (Boolean) function $f : 2^N \to \{0, 1\}$ with $f(\emptyset) = 0$ is called α-roughly weighted for an $\alpha \in \mathbb{R}_{\geq 1}$ if there are weights $w_1, \ldots, w_n \in \mathbb{R}$ fulfilling*

$$w(S) \geq 1 \qquad \forall S \subseteq N : f(S) = 1$$

and

$$w(S) \leq \alpha \qquad \forall S \subseteq N : f(S) = 0.$$

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[2]We remark that usually $f(\emptyset) = 1$ is possible for Boolean functions too. In our context the notion of α-roughly weightedness makes sense for $f(\emptyset) = 0$, so that we generally require this property. Later on, we specialize these sets to monotone Boolean functions with the additional restriction $f(N) = 1$, called simple games, and use the notation $\mathcal{S}_n$. Even more refined subclasses are the set $\mathcal{C}_n$ of complete simple games and the set $\mathcal{W}_n$ of weighted voting games on n voters. These sets are ordered as $\mathcal{B}_n \supseteq \mathcal{S}_n \supseteq \mathcal{C}_n \supseteq \mathcal{W}_n$, where the inclusions are strict if n is large enough. In order to state examples in a compact manner we often choose weighted voting games χ, since they can be represented by $[q; w_1, \ldots, w_n]$, where q is a quota and the w_i are weights. We have $\chi(S) = 1$ if and only if the sum $\sum_{i \in S} w_i \geq q$ for each subset $S \subseteq N$.

We remark that a function f with $f(\emptyset) = 1$ cannot be α-roughly weighted for any $\alpha \in \mathbb{R}$. In contrast to most definitions of roughly weighted games we allow negative weights, in the first run, and consider a wider class than simple games in our initial definition, i.e. Boolean functions with $f(\emptyset) = 0$. Later on, we will focus on subclasses of $\mathcal{B}_n$, where we can assume that all weights are non-negative. By $\mathcal{T}_\alpha$ (instead of $\mathcal{C}_\alpha$ as in [Gvozdeva et al., 2012]) we denote the class of all α-roughly weighted Boolean functions f with $f(\emptyset) = 0$. If $f \in \mathcal{T}_\alpha$ but $f \notin \mathcal{T}_{\alpha'}$ for all $1 \le \alpha' < \alpha$, we call α the critical threshold value $\mu(f)$ of f. Given f we can determine the critical threshold value using the following linear program:

(1)
$$\begin{aligned}
\text{Min} \quad & \alpha \\
& w(S) \ge 1 \quad \forall S \subseteq N : f(S) = 1 \\
& w(S) \le \alpha \quad \forall S \subseteq N : f(S) = 0 \\
& \alpha \ge 1 \\
& w_1, \dots, w_n \in \mathbb{R}
\end{aligned}$$

We consider it convenient to explicitly add the constraint $\alpha \ge 1$ in Definition 1, in accordance with the definition in [Gvozdeva et al., 2012], and in the linear program (1). Otherwise we would obtain the optimal solution $\alpha = 0$ for the weighted game $[2; 1, 1] \in \mathcal{B}_2$ or the optimal solution $\alpha = \frac{2}{3}$ for the weighted game $[3; 2, 2, 1, 1] \in \mathcal{B}_4$ using the weights $w_1 = w_2 = \frac{2}{3}$ and $w_3 = w_4 = \frac{1}{3}$. Since there are no coalitions with weights strictly between $\frac{2}{3}$ and 1 there are no contradicting implications. Arguably, values less than 1 contain more information, but on the other hand makes notation more complicated. To avoid any misconception we directly require $\alpha \ge 1$ (as in Definition 1) to guarantee non-contradicting implications independently from the possible weights of the coalitions.

At first, we remark that the inequality system (1) has at least one feasible solution given by $w_i = 1$ for all $1 \le i \le n$ and $\alpha = n$. Next we observe that the critical threshold value is a rational number, as it is the optimum solution of a linear programming problem with rational coefficients, and that we can restrict ourselves to rational weights w_i. For a general Boolean function $f : 2^N \to \{0, 1\}$ with $f(\emptyset) = 0$ negative weights may be necessary to achieve the critical threshold value. An example is given by the function f of three variables whose entire set of coalitions S with $f(S) = 1$ is given by $\{\{1\}, \{2\}, \{1, 2\}\}$. By considering the weights $w_1 = w_2 = 1$, $w_3 = -2$ we see that it is 1-roughly weighted. On the other hand we have the inequalities $w_1 \ge 1$, $w_2 \ge 1$, and $w_1 + w_2 + w_3 \le \alpha = 1$ from which we conclude $w_3 \le -1$. Another way to look at this example is to say that the critical threshold value would be 2 if only non-negative weights are allowed. (Here $n = 3$ voters are the smallest possibility, i.e. for $n \le 2$ there are non-negative realizations for the critical threshold value.)

A quite natural question is to ask for the largest critical threshold value $\mu(f)$ within the class of all Boolean functions $f : 2^N \to \{0, 1\}$ with $f(\emptyset) = 0$, which we denote by $c_{\mathcal{B}}(n)$, i.e. $c_{\mathcal{B}}(n) = \max\{\mu(f) \mid f \in \mathcal{B}_n\}$.

Lemma 1. $c_{\mathcal{B}}(n) = n$.

Proof. By choosing the weights $w_i = 1$ for all $1 \le i \le n$ we have $1 \le w(S) \le n$ for all $\emptyset \ne S \subseteq N$. Thus all functions $f : 2^N \to \{0, 1\}$ with $f(\emptyset) = 0$ are n-roughly weighted. The maximum $c_{\mathcal{B}}(n) = n$ is attained for example at the function with $f(N) = 0$ and $f(\{i\}) = 1$ for all $1 \le i \le n$. Since the singletons $\{i\}$ are winning, we have $w_i \ge 1$ for all $i \in N$, so that $w(N) \ge n$ while N is a losing coalition. $\qquad\square$

We would like to remark that if we additionally require $f(N) = 1$, then the critical threshold value is at most $n - 1$, which is tight (the proof of Lemma 1 can be easily adapted).

More interesting subclasses of Boolean functions with $f(\emptyset) = 0$ are simple games, i.e. monotone Boolean functions with $f(\emptyset) = 0$ and $f(N) = 1$, where $f(S) \le f(T)$ for all $S \subseteq T$. By $\mathcal{T}_\alpha \cap \mathcal{S}_n$ we denote the class of all α-roughly weighted simple games consisting of n voters and by $c_{\mathcal{S}}(n) := \max\{\mu(f) \mid f \in \mathcal{S}_n\}$ the largest critical threshold value within the class of simple games consisting of n voters. For simple games we can restrict ourselves to non-negative weights and can drop some of the inequalities in the linear program (1). (This is not true for general Boolean functions as demonstrated in the previous example.)

Lemma 2. *All simple games $\chi \in \mathcal{T}_\alpha \cap \mathcal{S}_n$ admit a representation in non-negative weights.*

Proof. Let $w_i \in \mathbb{R}$, for $1 \le i \le n$, be suitable weights. We set $w_i' := \max(w_i, 0) \in \mathbb{R}_{\ge 0}$ for all $1 \le i \le n$. For each winning coalition $S \subseteq N$ we have $w'(S) \ge w(S) \ge 1$. Due to the monotonicity property of simple games for each losing coalition $T \subseteq N$ the coalition $T' := \{i \in T : w_i \ge 0\}$ is also losing. Thus we have $w'(T) \le w(T') \le \alpha$. $\qquad\square$

We remark that we have not used $\chi(\emptyset) = 0$ or $\chi(N) = 1$ so that the statement can be slightly generalized.

Definition 2. *Given a simple game* χ *a coalition* $S \subseteq N$ *is called a* minimal winning coalition *if* $\chi(S) = 1$ *and* $\chi(S') = 0$ *for all proper subsets* S' *of* S. *Similarly, a coalition* $T \subseteq N$ *is called a* maximal losing coalition *if* $\chi(T) = 0$ *and* $\chi(T') = 1$ *for all* $T' \subseteq N$ *where* T *is a proper subset of* T'. *By* $\mathcal{W}$ *we denote the set of minimal winning coalitions and by* $\mathcal{L}$ *the set of maximal losing coalitions.*

We would like to remark that a simple game can be completely reconstructed from either the set $\mathcal{W}$ of its minimal winning coalitions or the set $\mathcal{L}$ of its maximal losing coalitions, i.e. a coalition $S \subseteq N$ is winning if and only if it contains a subset $S' \in \mathcal{W}$. Similarly, a coalition $T \subseteq N$ is losing if there is a $T' \in \mathcal{L}$ with $T \subseteq T'$.

Proposition 1. *The critical threshold value* $\mu(\chi)$ *of a simple game* $\chi \in \mathcal{S}_n$ *is given by the optimal target value of the following linear program:*

$$
\begin{aligned}
Min \quad & \alpha \\
& w(S) \geq 1 \quad \forall S \in \mathcal{W} \\
& w(S) \leq \alpha \quad \forall S \in \mathcal{L} \\
& \alpha \geq 1 \\
& w_1, \ldots, w_n \geq 0
\end{aligned}
$$

Proof. Due to Lemma 2 we can assume w.l.o.g. that $w_1, \ldots, w_n \geq 0$. With this it suffices to prove that a feasible solution of the stated linear program is also feasible for the linear program (1). Let $S \subseteq N$ be an arbitrary winning coalition, i.e., $\chi(S) = 1$. Since there exists an $S' \in \mathcal{W}$ with $S' \subseteq S$ we have

$$
w(S) \overset{w_i \geq 0}{\geq} w(S') \geq 1.
$$

Similarly, for each losing coalition $T \subseteq N$ there exists a $T' \in \mathcal{L}$ with $T \subseteq T'$ so that we have

$$
w(T) \overset{w_i \geq 0}{\leq} w(T') \leq \alpha.
$$

$\square$

Again, we have not used $\chi(\emptyset) = 0$ or $\chi(N) = 1$ in the proof.

A well studied subclass of simple games (and superclass of weighted voting games) arises from Isbell's desirability relation, see [Isbell, 1958]: We write $i \sqsupset j$ for two voters $i, j \in N$ iff we have $\chi\left(\{i\} \cup S \backslash \{j\}\right) \geq \chi(S)$ for all $j \in S \subseteq N \backslash \{i\}$. A pair (N, χ) is called a *complete simple game* if it is a simple game and the binary relation $\sqsupset$ is a total preorder. To factor out symmetry we assume $i \sqsupset j$ for all $1 \leq i < j \leq n$, i.e. voter i is at least as powerful as voter j, in the following. We abbreviate $i \sqsupset j$, $j \sqsupset i$ by $i \square j$ forming equivalence classes of voters $N_1, \ldots, N_t$. Let us denote $|N_i| = n_i$ for $1 \leq i \leq t$. We assume that those equivalence classes are ordered with decreasing influence, i.e. for $u \leq v$, $i \in N_u$, $j \in N_v$ we have $i \sqsupset j$. A coalition in a complete simple game can be described by the numbers a_h of voters from equivalence class N_h, i.e. by a vector $(a_1, \ldots, a_t)$. Note that the same vector represents $\binom{n_1}{a_1}\binom{n_2}{a_2} \ldots \binom{n_t}{a_t}$ coalitions that only differ in equivalent voters.

To transfer the concept of minimal winning coalitions and maximal losing coalitions to vectors, we need a suitable partial ordering:

Definition 3. *For two integer vectors* $\widetilde{a} = (a_1, \ldots, a_t)$ *and* $\widetilde{b} = (b_1, \ldots, b_t)$ *we write* $\widetilde{a} \preceq \widetilde{b}$ *if we have* $\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i$ *for all* $1 \leq k \leq t$. *For* $\widetilde{a} \preceq \widetilde{b}$ *and* $\widetilde{a} \neq \widetilde{b}$ *we use* $\widetilde{a} \prec \widetilde{b}$ *as an abbreviation. If neither* $\widetilde{a} \preceq \widetilde{b}$ *nor* $\widetilde{b} \preceq \widetilde{a}$ *holds we write* $\widetilde{a} \bowtie \widetilde{b}$.

In words, we say that $\widetilde{a}$ is smaller than $\widetilde{b}$ if $\widetilde{a} \prec \widetilde{b}$ and that $\widetilde{a}$ and $\widetilde{b}$ are incomparable if $\widetilde{a} \bowtie \widetilde{b}$.

With Definition 3 and the representation of coalitions as vectors in $\mathbb{N}^t$ at hand, we can define:

Definition 4. *A vector* $\widetilde{m} := (m_1, \ldots, m_t)$ *in a complete simple game* $\left((n_1, \ldots, n_t), \chi\right)$ *is a* shift-minimal winning vector *if* $\widetilde{m}$ *is a winning vector and every vector* $\widetilde{m}' \prec \widetilde{m}$ *is losing. Analogously, a vector* $\widetilde{m}$ *is a* shift-maximal losing vector *if* $\widetilde{m}$ *is a losing vector and every vector* $\widetilde{m}' \succ \widetilde{m}$ *is winning.*

As an example we consider the complete simple game $\chi \in \mathcal{C}_4$ whose minimal winning coalitions are given by $\{1,2\}, \{1,3\}, \{1,4\}$, and $\{2,3,4\}$. The equivalence classes of voters are given by $N_1 = \{1\}$ and $N_2 = \{2,3,4\}$. With this the shift-minimal winning vectors are given by $(1,1)$ and $(0,3)$. By $\overline{\mathcal{W}}$ we denote the set of shift-minimal winning vectors and by $\overline{\mathcal{L}}$ the set of shift-maximal losing vectors. Each complete simple game can be entirely reconstructed from either $\overline{\mathcal{W}}$ or $\overline{\mathcal{L}}$.

In [Carreras and Freixas, 1996] there is a very useful parameterization theorem for complete simple games:

Theorem 1.
(a) *Let a vector*
$$\widetilde{n} = (n_1, \ldots, n_t) \in \mathbb{N}_{>0}^t$$
and a matrix
$$\mathcal{M} = \begin{pmatrix} m_{1,1} & m_{1,2} & \ldots & m_{1,t} \\ m_{2,1} & m_{2,2} & \ldots & m_{2,t} \\ \vdots & \vdots & \ddots & \vdots \\ m_{r,1} & m_{r,2} & \ldots & m_{r,t} \end{pmatrix} = \begin{pmatrix} \widetilde{m}_1 \\ \widetilde{m}_2 \\ \vdots \\ \widetilde{m}_r \end{pmatrix}$$
be given, which satisfies the following properties:
  (i) $0 \le m_{i,j} \le n_j$, $m_{i,j} \in \mathbb{N}_{\ge 0}$ *for* $1 \le i \le r$, $1 \le j \le t$,
  (ii) $\widetilde{m}_i \bowtie \widetilde{m}_j$ *for all* $1 \le i < j \le r$,
  (iii) *for each* $1 \le j < t$ *there is at least one row-index* i *such that* $m_{i,j} > 0$, $m_{i,j+1} < n_{j+1}$ *if* $t > 1$ *and* $m_{1,1} > 0$ *if* $t = 1$, *and*
  (iv) $\widetilde{m}_i \succ \widetilde{m}_{i+1}$ *for* $1 \le i < t$ *(lexicographic order).*
  Then there exists a complete simple game (N, χ) *whose equivalence classes of voters have cardinalities as in* $\widetilde{n}$ *and whose shift-minimal winning vectors coincide with the rows of* $\mathcal{M}$.
(b) *Two complete games* $(\widetilde{n}_1, \mathcal{M}_1)$ *and* $(\widetilde{n}_2, \mathcal{M}_2)$ *are isomorphic, i.e., there exists a permutation of the voters so that the games are equal, if and only if* $\widetilde{n}_1 = \widetilde{n}_2$ *and* $\mathcal{M}_1 = \mathcal{M}_2$.

The rows of $\mathcal{M}$ correspond to the shift-minimal winning vectors whose number is denoted by r. The number of equivalence classes of voters is denoted by t.

By $c_{\mathcal{C}}(n) := \{\max \mu(\chi) \mid \chi \in \mathcal{C}_n\}$ we denote the largest critical threshold value within the class of complete simple games on n voters. As $\overline{\mathcal{W}} \subseteq \mathcal{W}$ and $\overline{\mathcal{L}} \subseteq \mathcal{L}$ we want to provide a linear programming formulation for the critical threshold value $\mu(\chi)$ of a complete simple game $\chi \in \mathcal{C}_n$, similar to Proposition 1, based on shift-minimal winning and shift-maximal losing vectors. At first, we show that we can further restrict the set of weights. To this end we call a feasible solution w of the inequality system in Proposition 1, where α is given, a *representation* (with respect to α).

Lemma 3. *All complete simple games* $\chi \in \mathcal{T}_\alpha \cap \mathcal{C}_n$ *admit a representation with weights satisfying* $w_1 \ge \cdots \ge w_n \ge 0$.

Proof. As $\chi \in \mathcal{C}_n \subseteq \mathcal{S}_n$ is a simple game, there exists a representation with weights $w'_1, \ldots, w'_n \in \mathbb{R}_{\ge 0}$ due to Lemma 2. Let (j, h) be the lexicographically smallest pair such that $w'_j < w'_h$ and $j < h$. By τ we denote the transposition (j, h), i.e. the permutation that swaps j and h, and set $w_i := w'_{\tau(i)}$.

For a winning coalition S with $j \in S$, $h \notin S$ we have $w(S) \ge w'(S) \ge 1$. If S is a winning coalition with $j \notin S$, $h \in S$ then $\tau(S)$ is a winning coalition too and we have $w(S) = w'(\tau(S)) \ge 1$. For a losing coalition T with $j \notin T$, $h \in T$ we have $w(T) \le w'(T) \le \alpha$. If T is a losing coalition with $j \in T$, $h \notin T$ then $\tau(T)$ is a losing coalition too and we have $w(T) = w'(\tau(T)) \le \alpha$.

By recursively applying this argument we can construct representing weights fulfilling $w_1 \ge \cdots \ge w_n \ge 0$. $\square$

We remark that the previous complete simple game with minimal winning coalitions $\{1,2\}, \{1,3\}, \{1,4\}$, and $\{2,3,4\}$ can be represented as a weighted voting game $[4; 3, 2, 1, 1]$. Another representation of the same game using equal weights for equivalent voters would be $[3; 2, 1, 1, 1]$.

Lemma 4. *All complete simple games* $\chi \in \mathcal{T}_\alpha \cap \mathcal{C}_n$ *admit a representation with weights* $w_1 \ge \cdots \ge w_n \ge 0$ *where voters of the same equivalence class have the same weight.*

Proof. Let $w'_1 \ge \cdots \ge w'_n \ge 0$ be a representation of χ and $N_1, \ldots, N_t$ the set of equivalence classes of voters. By $1 \le j \le t$ we denote the smallest index such that not all voters in N_j have the same weight and define new weights $w_i := w'_i$ for all $i \in N \backslash N_j$ and $w_i := \frac{\sum\limits_{h \in N_j} w'_h}{|N_j|}$, i.e. the arithmetic mean of the previous weights in N_j.

By recursively applying this construction we obtain a representation with the desired properties. It remains to show that the new weights w_i fulfill the α-conditions.

Let S be a winning coalition with $k = |S \cap N_j|$. By S' we denote the union of $S \backslash N_j$ and the k lightest voters from N_j. Since S' is a winning coalition too we have $w(S) \geq w'(S') \geq 1$. Similarly, let T be a losing coalition with $k = |T \cap N_j|$: By T' we denote the union of $T \backslash N_j$ and the k heaviest voters from N_j. Since T' is also a losing coalition we have $w(T) \leq w'(T') \leq \alpha$. $\square$

Lemma 5. *The critical threshold value $\mu(\chi)$ of a complete simple game $\chi \in \mathcal{C}_n$ with t equivalence classes of voters is given by the optimal target value of the following linear program:*

$$\begin{aligned}
\textit{Min} \quad & \alpha \\
& \sum_{i=1}^{t} a_i w_i \geq 1 \quad \forall (a_1, \cdots, a_t) \in \overline{\mathcal{W}} \\
& \sum_{i=1}^{t} a_i w_i \leq \alpha \quad \forall (a_1, \cdots, a_t) \in \overline{\mathcal{L}} \\
& \alpha \geq 1 \\
& w_i \geq w_{i+1} \quad \forall 1 \leq i \leq t-1 \\
& w_t \geq 0
\end{aligned}$$

Proof. Due to Lemma 4 we can assume that for the critical threshold value $\mu(\chi) = \alpha$ there exists a feasible weighting fulfilling the conditions of the stated linear program. It remains to show that $w(W) \geq 1$ and $w(L) \leq \alpha$ holds for all shift-winning vectors W and all losing vectors L. Therefore, we denote by $W' \in \overline{\mathcal{W}}$ an arbitrary shift-minimal winning vector with $W \succeq W'$ and by $L' \in \overline{\mathcal{L}}$ an arbitrary shift-maximal losing vector with $L \preceq L'$. The proof is finished by checking $w(L) \leq w(L') \leq \alpha$ and $w(W) \geq w(W') \geq 1$. $\square$

So, for complete simple games the number of constraints could be further reduced. In this context we remark that by additionally disregarding the conditions $w_i \geq w_{i+1}$ from the linear program we would lose the information about the order on equivalence classes. This effect is demonstrated by the following example. Let us consider the complete simple game $(n_1, n_2) = (15, 4)$ with unique shift-minimal winning vector $(7, 2)$. There are two shift-maximal losing vectors: $(8, 0)$ and $(6, 4)$. Choosing the special solution $w_1 = \frac{1}{14}$, $w_2 = \frac{1}{4}$, $\alpha = \frac{3}{2}$ would be feasible for

$$\begin{aligned}
7w_1 + 2w_2 &\geq 1 \\
8w_1 &\leq \alpha \\
6w_1 + 4w_2 &\leq \alpha \\
\alpha &\geq 1 \\
w_1, w_2 &\geq 0
\end{aligned}$$

For the coalition $(8, 1)$ we obtain the weight $8w_1 + 1w_2 = \frac{23}{28} < 1$, so that it should be a losing coalition, which is a contradiction to $(8, 1) \succeq (7, 2)$. So we have to use the ordering on the weights.

At the beginning of this section we have argued that the condition $\alpha \geq 1$ is necessary, since otherwise the optimal target value of the stated linear programming formulations will not coincide with $\mu(\chi)$ in all cases. On the other hand, if $z^\star(\chi)$ denotes the optimal target value of one of the stated LPs, where we have dropped the condition $\alpha \geq 1$, then we have

$$\mu(\chi) = \max(z^\star(\chi), 1).$$

In the following we will drop the condition $\alpha \geq 1$ whenever it seems beneficial for the ease of a shorter presentation while having the just mentioned exact correspondence in mind.

An important solution concept in cooperative game theory is the *core*, i.e. the set of all stable imputations, see e.g. [Tijs, 2011] for an introduction. Since the core can be empty under certain circumstances, the possibility of external payments was considered in order to stabilize the outcome, see [Bachrach et al., 2009]. The external party quite naturally is interested in minimizing its expenditures. This leads to the concept of the *cost of stability* (CoS) of a coalition game. Skipping the relation of CoS with the core, we directly define the cost of stability $CoS(f)$ of

a given Boolean function f with $f(\emptyset) = 0$ as the solution of the following linear program:

$$(2) \qquad\qquad \text{Min} \qquad \Delta$$

$$(3) \qquad\qquad \Delta \;\geq\; 0$$

$$(4) \qquad\qquad \sum_{i \in N} p_i \;=\; f(N) + \Delta$$

$$(5) \qquad\qquad \sum_{i \in S} p_i \;\geq\; f(S) \quad \forall S \subseteq N$$

$$(6) \qquad\qquad p_i \;\geq\; 0 \qquad \forall i \in N.$$

The cost of stability is an upper bound for the critical threshold value:

Lemma 6. *For a Boolean function $f \in \mathcal{B}_n$ with $f(N) = 1$ we have $\mu(f) \leq 1 + CoS(f)$.*

Proof. Let $p_1, \ldots, p_n, \Delta$ be an optimal solution for the above linear program for the cost of stability. If we choose the weights as $w_i = p_i$, then we have $w_i \in \mathbb{R}$ and we have $w(S) \geq 1$ for all winning coalitions S due to constraint (5). Applying constraint (6) and constraint (4) yields

$$w(S) = \sum_{i \in S} p_i \leq \sum_{i \in N} p_i = f(N) + \Delta = 1 + CoS(f)$$

for all coalitions $S \subseteq N$. Thus every losing coalition has a weight of at most $1 + CoS(f)$. $\qquad\square$

Due to $CoS(f) \leq n \cdot \max_{S \subseteq N} f(S) \leq n$, see Theorem 3.4 in [Bachrach et al., 2009], we have $CoS(f) \leq n$ for all $f \in \mathcal{B}_n$, where equality is attained for the Boolean function with $f(S) = 1$ for all $S \neq \emptyset$. With respect to Lemma 1 we mention the relation

$$c_{\mathcal{B}}(n) = \max_{f \in \mathcal{B}_n} \mu(f) = \max_{f \in \mathcal{B}_n} CoS(f) = n.$$

On the other hand, we observe that the ratio between $CoS(f)$ and $\mu(f)$ can be quite large. Theorem 3.3 in [Bachrach et al., 2009] states $CoS(\chi) = \frac{n}{|q|} - 1$ for the weighted voting game $\chi = [q; w, \ldots, w]$, while we have $\mu(\chi) = 1$. Setting $w = q = 1$ we see that the ration can become at least as large as $n - 1$.

By imposing more structure on the set of feasible games, the bound $CoS(f) \leq n$, for $f \in \mathcal{B}_n$, could be reduced significantly. To this end we introduce further notation:

Definition 5. *A Boolean function $f \in \mathcal{B}_n$ is called* super-additive *if we have $f(S) + f(T) \leq f(S \cup T)$ for all disjoint coalitions $S, T \subseteq N$. It is called* anonymous *if we have $f(S) = f(T)$ for all coalitions $S, T \subseteq N$ with $|S| = |T|$, i.e. the outcome only depends on the cardinality of the coalition.*

In our context super-additivity means that each pair of winning coalitions has a non-empty intersection, which is also called a *proper* game. These are the most used voting games for real world institutions.

3. CERTIFICATES

In computer science, more precisely in complexity theory, a certificate is a string that certifies the answer to a membership question (or the optimality of a computed solution). In our context we e.g. want to know whether a given simple game $\chi \in \mathcal{S}_n$ is α-roughly weighted. If the answer is yes, we just need to state suitable weights. Given the weights, the answer then can be checked by testing the validity of the inequalities in the linear program of Proposition 1. Since both $\mathcal{W}$ and $\mathcal{L}$ form antichains, i.e. no element is contained in another, we can conclude from Sperner's theorem that at most $2\binom{n}{\lfloor n/2 \rfloor} + n + 1$ inequalities have to be checked. But also in the other case, where the answer is no, we would like to have a computational witness that χ is not α-roughly weighted.

For weighted voting games trading transforms, see e.g. [Taylor and Zwicker, 1999], can serve as a certificate for non-weightedness. In [Gvozdeva and Slinko, 2011] this concept has been transferred to roughly weighted games and it was proven that for each non-weighted simple game consisting of n voters there exists a trading transform of length at most $\left\lfloor (n+1) \cdot 2^{\frac{1}{2} n \log_2 n} \right\rfloor$.

Using the concept of duality in linear programming one can easily construct a certificate for the fact that a given voting structure χ is not α'-roughly weighed for all $\alpha' < \alpha$, where $\alpha \geq 1$ is fixed. To be more precise, we present a certificate for the inequality $\mu(\chi) \geq \alpha$.

The dual of a general linear program $\min c^T x$, $Ax \geq b$, $x \geq 0$ (called primal) is given by $\max b^T y$, $A^T y \leq c$, $y \geq 0$. The strong duality theorem, see e.g. [Vanderbei, 2008], states that if the primal has an optimal solution, $x^\star$, then the dual also has an optimal solution, $y^\star$, such that $c^T x^\star = b^T y^\star$. As mentioned before, the linear program

for the determination of the critical threshold value always has an optimal solution, so that we can apply the strong duality theorem to obtain a certificate.

Considering only a subset of the winning coalitions for the determination of the critical threshold value means removing some constraints of the corresponding linear program. This enlarges the feasible set such that the optimal solution will eventually decrease but not increase. For further utilization we state the resulting lower bound for the critical threshold value of this approach:

Lemma 7. *For a given simple game $\chi \in \mathcal{S}_n$ let W' be a subset of its winning coalitions and L' be a subset of its losing coalitions. If (u, v) is a feasible solution of the following linear program with target value α' then we have $\mu(\chi) \geq \alpha'$.*

$$
\begin{aligned}
Max \quad & \sum_{S \in W'} u_S \\
& \sum_{S \in W' : i \in S} u_S - \sum_{T \in L' : i \in T} v_T \leq 0 \quad \forall 1 \leq i \leq n \\
& \sum_{T \in L'} v_T \leq 1 \\
& u_S \geq 0 \quad \forall S \in W' \\
& v_T \geq 0 \quad \forall T \in L'
\end{aligned}
$$

Proof. The stated linear program is the dual of

$$
\begin{aligned}
Min \quad & \alpha \\
& \sum_{i \in S} w_i \geq 1 \quad \forall S \in W' \\
& \alpha - \sum_{i \in T} w_i \geq 0 \quad \forall T \in L' \\
& w_i \geq 0 \quad \forall 1 \leq i \leq n,
\end{aligned}
$$

which is a relaxation of the linear program (1) determining the critical threshold value. $\qquad\square$

To briefly motivate the underlying ideas we consider an example. Let the simple game χ for 5 voters be defined by its set $\Big\{ \{1, 2\}, \{2, 4\}, \{3, 4\}, \{2, 5\}, \{3, 5\} \Big\}$ of minimal winning coalitions. The set of maximal losing coalitions is given by $\Big\{ \{1, 3\}, \{2, 3\}, \{1, 4, 5\} \Big\}$. For this example the linear program of Proposition 1 to determine the critical α (after some easy equivalence transformations) reads as

$$
\begin{aligned}
Min \quad & \alpha \quad \text{s.t.} \\
& w_1 + w_2 \geq 1 \\
& w_2 + w_4 \geq 1 \\
& w_3 + w_4 \geq 1 \\
& w_2 + w_5 \geq 1 \\
& w_3 + w_5 \geq 1 \\
& \alpha - w_1 - w_3 \geq 0 \\
& \alpha - w_2 - w_3 \geq 0 \\
& \alpha - w_1 - w_4 - w_5 \geq 0 \\
& \alpha \geq 1 \\
& w_1 \geq 0, \ldots, w_5 \geq 0
\end{aligned}
$$

(We have replaced the conditions $w(S) \leq \alpha$ for the losing coalitions S by $\alpha - w(S) \geq 0$.)

Running a linear program solver yields the optimal solution $w_1 = w_4 = w_5 = \frac{2}{5}$, $w_2 = w_3 = \frac{3}{5}$, and $\alpha = \frac{6}{5}$. By inserting these values into the inequalities of the stated linear program we can check that $\chi \in \mathcal{T}_{\frac{6}{5}} \cap \mathcal{S}_5$. Thus the weights form a certificate for this fact.

To obtain a certificate for the fact that $\chi \notin \mathcal{T}_{\alpha'}$ for all $\alpha' < \frac{6}{5}$, i.e. $\mu(\chi) \geq \frac{6}{5}$, we consider the dual problem:

$$
\begin{aligned}
Max \quad & y_1 + y_2 + y_3 + y_4 + y_5 + z \quad \text{s.t.} \\
& y_1 - y_6 - y_8 \leq 0 \\
& y_1 + y_2 + y_4 - y_7 \leq 0 \\
& y_3 + y_5 - y_7 \leq 0 \\
& y_2 + y_3 - y_8 \leq 0 \\
& y_4 + y_5 - y_8 \leq 0 \\
& y_6 + y_7 + y_8 + z \leq 1 \\
& y_1 \geq 0, \ldots, y_8, z \geq 0
\end{aligned}
$$

An optimal solution is given by $y_1 = y_5 = y_8 = \frac{2}{5}$, $y_2 = y_3 = \frac{1}{5}$, $y_7 = \frac{3}{5}$, and $y_4 = y_6 = z = 0$ with target value $\frac{6}{5}$ (as expected using the strong duality theorem). In combination with the weak duality theorem, see e.g. [Vanderbei, 2008], the stated feasible dual solution (y, z) is a certificate for the fact that the critical threshold value for the simple game χ is larger or equal to $\frac{6}{5}$. In general, the optimal solution vector (y, z) has at most $n + 1$ non-zero entries so that we obtain a very short certificate.

We would like to remark that one can use the values of the dual variables as multipliers for the inequalities in the primal problem to obtain the desired bound on the critical threshold value. In our case multiplying all inequalities with the respective values yields

$$\frac{2}{5} \cdot (w_1 + w_2) + \frac{1}{5} \cdot (w_2 + w_4) + \frac{1}{5} \cdot (w_3 + w_4) + 0 \cdot (w_2 + w_5) + \frac{2}{5} \cdot (w_3 + w_5)$$

$$+0 \cdot (\alpha - w_1 - w_3) + \frac{3}{5} \cdot (\alpha - w_2 - w_3) + \frac{2}{5} \cdot (\alpha - w_1 - w_4 - w_5) + 0 \cdot \alpha$$

$$\geq \frac{2}{5} + \frac{1}{5} + \frac{1}{5} + 0 + \frac{2}{5} + 0 = \frac{6}{5}$$

which is equivalent to $\alpha \geq \frac{6}{5}$, i.e. a certificate for the fact that $\chi \notin \mathcal{T}_{\alpha'} \cap \mathcal{S}_5$ for $\alpha' < \frac{6}{5}$.

4. MAXIMAL CRITICAL THRESHOLD VALUES

In Lemma 1 we have shown that the maximum critical threshold value of a Boolean function $f : 2^N \to \{0, 1\}$ with $f(\emptyset) = 0$ is given by $c_{\mathcal{B}}(n) = n$. If additionally $f(N) = 1$ is required the upper bound drops to $n - 1$ (which is tight). In this section, we want to provide bounds for the maximal critical threshold values for simple games and complete simple games on n voters. By considering a complete simple game with two types of voters we can derive a lower bound of $\Omega(\sqrt{n})$ for $c_{\mathcal{C}}(n)$. Apart from constants, this bound is conjectured to be tight. This will be substantiated by upper bounds of $O(\sqrt{n})$ for $c_{\mathcal{C}}(n)$ for several special subclasses of complete simple games. For the general case, we can only obtain the result that $c_{\mathcal{C}}(n)$ is asymptotically smaller than $O(n)$, which is the asymptotic of the maximum critical threshold value for simple games. Finally, we relate the more sophisticated upper bounds on the cost of stability from [Bachrach et al., 2009] to upper bounds for the critical threshold value for other special subclasses of Boolean games.

The authors of [Gvozdeva et al., 2012] have proven the bounds $\frac{1}{2} \lfloor \frac{n}{2} \rfloor \leq c_{\mathcal{S}}(n) \leq \frac{n-2}{2}$ for $n \geq 4$ and determined the exact values $c_{\mathcal{S}}(1) = c_{\mathcal{S}}(2) = c_{\mathcal{S}}(3) = c_{\mathcal{S}}(4) = 1$, $c_{\mathcal{S}}(5) = \frac{6}{5}$, $c_{\mathcal{S}}(6) = \frac{3}{2}$. By considering null voters we conclude $c_{\mathcal{S}}(n) \leq c_{\mathcal{S}}(n + 1)$ and $c_{\mathcal{C}}(n) \leq c_{\mathcal{C}}(n + 1)$ for all $n \in \mathbb{N}$.

Proposition 2. *For $n \geq 4$ we have $c_{\mathcal{S}}(n) \geq \frac{\lfloor \frac{n^2}{4} \rfloor}{n}$.*

Proof. For the even integers we took an example from [Gvozdeva et al., 2012] and consider for $n = 2k$ the simple game uniquely defined by the minimal winning coalitions $W_i = \{2i - 1, 2i\}$ for $1 \leq i \leq k$. Then the two coalitions $L_1 = \{1, 3, \ldots, 2k - 1\}$ and $L_2 = \{2, 4, \ldots, 2k\}$ are maximal losing coalitions. Our example given above is of this type ($k = 4$). We apply Lemma 7 with $u_{W_1} = \cdots = u_{W_k} = v_{L_1} = v_{L_2} = \frac{1}{2}$ to deduce $c_{\mathcal{S}}(n) \geq \sum_{i=1}^{k} \frac{1}{2} = \frac{n}{4}$.

Using a null voter, as done in [Gvozdeva et al., 2012], gives $c_{\mathcal{S}}(n) \geq \frac{n-1}{4}$ for odd n, where $\frac{\lfloor \frac{n^2}{4} \rfloor}{n} - \frac{n-1}{4} = \frac{n-1}{4n}$.

For odd $n = 2k + 1$ we consider the simple game uniquely defined by the minimal winning coalitions $W_i = \{i, i + 1\}$ for $1 \leq i \leq n - 1$. Two maximal losing coalitions are given by $L_1 = \{1, 3, \ldots, 2k + 1\}$ and $L_2 = \{2, 4, \ldots, 2k\}$. Next we apply Lemma 7 and construct a certificate for $c_{\mathcal{S}}(n) \geq \frac{(n-1)(n+1)}{4n} = \frac{\lfloor \frac{n^2}{4} \rfloor}{n}$. We set $u_{W_{2i-1}} = \frac{k+1-i}{n}$, $u_{W_{2i}} = \frac{i}{n}$ for all $1 \leq i \leq k$, $v_{L_1} = \frac{k}{n}$, $v_{L_2} = \frac{k+1}{n}$ and check that it is a feasible solution. Since $\sum_{i=1}^{n-1} u_{W_i} = \frac{k(k+1)}{n} = \frac{(n-1)(n+1)}{4n}$ the proposed lower bound follows. $\square$

So, we are only able to slightly improve the previously known lower bound for $c_{\mathcal{S}}(n)$ if the number of voters is odd. One can easily verify that the given examples have a critical threshold value of $\frac{\lfloor \frac{n^2}{4} \rfloor}{n}$.

Conjecture 1. *For $n \geq 4$ we have $c_{\mathcal{S}}(n) = \frac{\lfloor \frac{n^2}{4} \rfloor}{n}$.*

We would like to remark that the simple game defined in the proof of Proposition 2 is very far from being the unique one with $\mu(\chi) = \frac{\left\lfloor \frac{n^2}{4} \right\rfloor}{n}$. For the proof we need that L_1, L_2 are losing coalitions and that the stated subsets of cardinality two are winning coalitions. We can construct an exponential number of simple games having a critical α of at least $\frac{\left\lfloor \frac{n^2}{4} \right\rfloor}{n}$ as follows: Let $L_1' \subsetneq L_1$ and $L_2' \subsetneq L_2$ such that none of the winning coalitions of size two is contained in $L_1' \cup L_2'$ and $|L_1'|, |L_2'| \geq 1$. With this we can specify the coalition $L_1' \cup L_2'$ either as winning or as losing without violating the other properties. This fact suggests that it might be hard to solve the integer linear program exactly to determine $c_{\mathcal{S}}(n)$ for larger values of n, see Section 5.

Another concept to measure the deviation of a simple game χ from a weighted voting game is its dimension, i.e. the smallest number k of weighted voting games that χ is given by their intersection, see e.g. [Deĭneko and Woeginger, 2006]. It is well known that each simple game has a finite dimension (depending on n), see [Taylor and Zwicker, 1993]. Simple games of dimension 1 coincide with weighted voting games having a critical threshold value of 1. The next possible dimension is two, where the critical threshold can be as large as the best known lower bound of $\left\lfloor \frac{n^2}{4} \right\rfloor / n$. Thus, there is no direct relation between the dimension of a simple game and its critical threshold value. To construct such examples we split the voters into sets of cardinality of at least $\left\lfloor \frac{n}{2} \right\rfloor$, i.e. as uniformly distributed as possible, and assign weight vectors $(1, 0)$ to the elements of one such set and $(0, 1)$ to the elements from the other set. Using a quota vector $(1, 1)$ we obtain a simple game that satisfies the necessary requirements for a critical α of at least $\left\lfloor \frac{n^2}{4} \right\rfloor / n$. In other words the dimension of a simple game is somewhat independent from the critical threshold parameter.

Lemma 8. *Let χ be a simple game with n voters and $\mu(\chi) = \alpha$. If a losing coalition of cardinality k exists, then we have $\alpha \leq n - k$.*

Proof. Let $S \subsetneq N$ be a losing coalition of cardinality k. We use the weights $w_i = 0$ for all $i \in S$ and $w_i = 1$ for all $i \in N \backslash S$. Since $w(N) = n - k$ the weight of each losing coalition is at most $n - k$ and since each winning coalition must contain at least one element from $N \backslash S$ their weight is at least 1. $\square$

Lemma 9. *Let χ be a simple game with n voters and $\mu(\chi) = \alpha$. If the maximum size of a losing coalition is denoted by k we have $\alpha \leq \max\left(1, \frac{k}{2}\right)$.*

Proof. We assign a weight of 1 to every voter i where $\{i\}$ is a winning coalition and a weight of $\frac{1}{2}$ to every other voter. Thus each winning coalition has a weight of at least 1 and each losing coalition a weight of at most $\frac{k}{2}$. $\square$

Corollary 1. *For each integer $n \geq 3$ we have $c_{\mathcal{S}}(n) \leq \frac{n}{3}$.*

Proof. Let χ be a simple game with largest losing coalition of size k and consisting of n voters. If $k \leq \frac{2n}{3}$ then we have $\mu(\chi) \leq \max\left(1, \frac{k}{2}\right) \leq \frac{n}{3}$. Otherwise, we have $\mu(\chi) \leq n - k \leq \frac{n}{3}$. $\square$

To further improve Corollary 1 some reduction techniques might be useful.

Lemma 10. *If a simple game χ on $n \geq 2$ voters contains a winning coalition of cardinality one then we have $\mu(\chi) \leq c_{\mathcal{S}}(n-1)$.*

Proof. W.l.o.g. let $\{n\}$ be a winning coalition. If $\{1, \ldots, n-1\}$ is a losing coalition then χ is roughly weighted using the weights $w_1 = \cdots = w_{n-1} = 0$, $w_n = 1$. Otherwise we consider the simple game χ' arising from χ by dropping voter n. Let $w_1, \ldots, w_{n-1}$ be a weighting for χ' corresponding to a threshold value of at most $c_{\mathcal{S}}(n-1)$. By choosing $w_n = 1$ we can extend this to a valid weighting for χ since every coalition which contains voter n is a winning coalition. $\square$

From now on, we consider complete simple games. To provide a lower bound on $c_{\mathcal{C}}(n)$ we consider a special subclass of complete simple games, i.e., complete simple games with $t = 2$ types of voters and a unique shift-minimal winning vector (a, b) $(r = 1)$. So, if a coalition contains at least a voters of the first type and and least $a + b$ members in total, then it is winning, otherwise it is losing.

In the following we will derive conditions on the parameters a and b in order to exclude weighted games, which would lead to a critical threshold value of 1. Since the shift-maximal losing vectors depend on a certain relation between a and b, we have to consider two different cases to state the linear program to determine the critical threshold value.

For $a+b-1 \leq n_1$ (case 1) the shift-maximal losing vectors are given by $(a+b-1, 0)$, $(a-1, n_2)$ and otherwise (case 2) by $(n_1, a+b-1-n_1)$, $(a-1, n_2)$.

Due to condition (a).(iii) in Theorem 1 we have $a > 0$. and $w_1 = w_2 = \frac{1}{b}$ shows that the game is roughly weighted in this case. For $a = n_1$ a quota of $q = n_1 n_2 + b$ and weights $w_1 = n_2$ and $w_2 = 1$ testify that the game is weighted. So, we only need to consider $1 \leq a \leq n_1 - 1$, $0 \leq b \leq n_2 - 1$. For $b = 0$ the games are weighted via quota $q = a$ and weights $w_1 = 1$, $w_2 = 0$. For $b = 1$ the games are weighted via quota $q = an_2 + 1$ and weights $w_1 = n_2$, $w_2 = 1$. If $b = n_2$ a quota of $q = a + n_2 - 1 + \frac{a}{n_1+n_2}$ and weights of $w_1 = 1 + \frac{1}{n_1+n_2}$, $w_2 = 1$ show that these games are weighted so that we can assume $2 \leq b \leq n_2 - 2$ and $n \geq 6$.

To compute $c_{\mathcal{C}}(n, r = 1, t = 2)$ we have to solve the linear program

$$\min \alpha \quad \text{s.t.}$$

$$(7) \qquad aw_1 + bw_2 \geq 1$$

$$(8) \qquad \alpha - (a+b-1)w_1 \geq 0$$

$$(9) \qquad \alpha - (a-1)w_1 - n_2 w_2 \geq 0$$

$$(10) \qquad w_1 \geq w_2$$

$$(11) \qquad w_2 \geq 0$$

for case 1 and

$$\min \alpha \quad \text{s.t.}$$

$$(12) \qquad aw_1 + bw_2 \geq 1$$

$$(13) \qquad \alpha - n_1 w_1 - (a+b-1-n_1)w_2 \geq 0$$

$$(14) \qquad \alpha - (a-1)w_1 - n_2 w_2 \geq 0$$

$$(15) \qquad w_1 \geq w_2$$

$$(16) \qquad w_2 \geq 0$$

for case 2. We would like to remark that we may also include the constraint $\alpha \geq 1$. Once it is tight we have $\alpha = 1$, so that we assume $\alpha > 1$ in the following.

The optimal solution of these linear programs is attained at a corner of the corresponding polytope which is the solution of a 3-by-3-equation system arising by combining three of the five inequalities. As notation we use $A \subset \{7, 8, 9, 10, 11\}$ with $|A| = 3$. (Some of these solutions may be infeasible.) At first, we remark that $w_1 = w_2 = 0$ is infeasible in both cases so that we assume $|A \cap \{10, 11\}| \leq 1$.

For case 1 the basic solutions, parameterized by sets of tight inequalities, are given by:

$\{7, 8, 9\}$ $w_1 = \frac{n_2}{an_2 + b^2}$, $w_2 = \frac{b}{an_2 + b^2}$, $\alpha = \frac{n_2(a+b-1)}{an_2 + b^2}$, always feasible, e.g. we have $n_2(b-1) \geq b^2$ due to $b \leq n_2 - 2$ and $b \geq 2$ so that $\alpha \geq 1$ holds.

$\{7, 8, 10\}$ $\alpha = \frac{a+b-1}{a+b} < 1$, contradiction

$\{7, 8, 11\}$ $w_1 = \frac{1}{a}$, $w_2 = 0$, $\alpha = \frac{a+b-1}{a}$, always feasible

$\{7, 9, 10\}$ $w_1 = \frac{1}{a+b}$, $w_2 = \frac{1}{a+b}$, $\alpha = \frac{a-1+n_2}{a+b}$, always feasible

$\{7, 9, 11\}$ $\alpha = \frac{a-1}{a} < 1$, contradiction

$\{8, 9, 10\}$ $\alpha = 0 < 1$, contradiction

$\{8, 9, 11\}$ $\alpha = 0 < 1$, contradiction

We always have $\frac{a+b-1}{a} > \frac{a+b-1}{a+\frac{b^2}{n_2}} = \frac{n_2(a+b-1)}{an_2 + b^2}$ and

$$(a+b) \cdot (an_2 + b^2) \cdot \left(\frac{a-1+n_2}{a+b} - \frac{n_2(a+b-1)}{an_2 + b^2} \right) = b(n_2 - b) + a(n_2 - b)^2 > 0.$$

Thus $\alpha = \frac{n_2(a+b-1)}{an_2 + b^2}$ is always the minimum value.

For case 2 the basic solutions are given by:

$\{12, 13, 14\}$ $w_1 = \frac{n_1 + n_2 + 1 - a - b}{-a^2 - 2ab + a + an_1 + n_1 b + an_2 + b}$, $w_2 = \frac{n_1 + 1 - a}{-a^2 - 2ab + a + an_1 + n_1 b + an_2 + b}$,

$\alpha = \frac{n_1 n_2 - ab + b - a^2 + 2a + an_1 - 1 - n_1}{-a^2 - 2ab + a + an_1 + n_1 b + an_2 + b} =: \alpha'$, where we have $w_1 \geq w_2$. $\alpha \geq 1$ is equivalent to $n_1 n_2 + a - 1 - n_1 \geq -ab + n_1 b + an_2$ which can be simplified to the valid inequality $\underbrace{(n_1 - a)}_{\geq 1} \cdot \underbrace{(n_2 - b - 1)}_{\geq 1} \geq 1$.

$\{12, 13, 15\}$ $\alpha = \frac{a+b-1}{a+b} < 1$, contradiction

$\{12, 13, 16\}$ $w_1 = \frac{1}{a}$, $w_2 = 0$, $\alpha = \frac{n_1}{a}$, always feasible

$\{12, 14, 15\}$ $w_1 = \frac{1}{a+b}$, $w_2 = \frac{1}{a+b}$, $\alpha = \frac{a-1+n_2}{a+b}$, always feasible

$\{12, 14, 16\}$ $\alpha = \frac{a-1}{a} < 1$, contradiction

$\{13, 14, 15\}$ $\alpha = 0 < 1$, contradiction

$\{13, 14, 16\}$ $\alpha = 0 < 1$, contradiction

$\alpha' \leq \frac{n_1}{a}$ is equivalent to

$$\frac{(n_1 + 1 - a) \cdot (a(n_1 + 1 - a) + b(n_1 - a))}{a \cdot (a(n_1 + n_2 + 1 - a - b) + b(n_1 + 1 - a))} \geq 0$$

and $\alpha' \leq \frac{a-1+n_2}{a+b}$ is equivalent to

$$\frac{(n_2 - b)(a(n_2 - b) + b)}{(a + b) \cdot \left(a(n_2 - b) + (a + b)(n_1 + 1 - a)\right)} \geq 0.$$

Since in both cases all factors are non-negative the respective inequalities are valid and the minimum possible α-value is given by α'.

To answer the question for the maximum possible α in case 1 depending on n we have to solve the following optimization problem

$$\max \frac{a + b - 1}{a + \frac{b^2}{n_2}} \quad \text{s.t.}$$
$$a + b - 1 \leq n_1$$
$$n_1 + n_2 = n$$
$$n_1, n_2 \geq 1$$
$$1 \leq a \leq n_1 - 1$$
$$2 \leq b \leq n_2 - 2,$$

where all variables have to be integers. For $z \geq 1$, $x > y > 0$ we have $\frac{z-1+x}{z-1+y} > \frac{z+x}{z+y}$. Thus the maximum is attained at the minimum value of a which is 1. ($a = 1$ also yields the weakest constraint $a + b - 1 \leq n_1$.) Since $1 \leq a \leq n_1 - 1$ is equivalent to $n_1 \geq 2$, which is implied by $a + b - 1 \leq n_1$ via $b \geq 2$, we can drop this constraint.

If $a + b - 1 < n_1$ then we could decrease n_1 by 1 and increase n_2 by 1 yielding a larger target value. Thus we have $a + b - 1 = n_1$, which is equivalent to $b = n_1$. Using $n_1 + n_2 = n$ yields $n_2 = n - b$. Inserting then yields the optimization problem

$$\max \frac{b}{1 + \frac{b^2}{n-b}}, \quad 2 \leq b \leq \frac{n-2}{2},$$

where $b, n \in \mathbb{N}$. Relaxing the integrality constraint results in

$$b = \left(\sqrt{n} - 1\right) \cdot \frac{n}{n - 1}$$

with optimal value

$$\frac{n^{5/2} - 2n^2 + n^{3/2}}{2n^2 - 3n^{3/2} + n^{1/2}} \leq \frac{\sqrt{n}}{2}$$

tending to $\frac{\sqrt{n}}{2}$ as n approaches infinity. Since the target function is continuous and there is only one inner local maximum, the optimal integer solution is either $b = \left\lfloor (\sqrt{n} - 1) \cdot \frac{n}{n-1} \right\rfloor$ or $b = \left\lceil (\sqrt{n} - 1) \cdot \frac{n}{n-1} \right\rceil$. For $n \geq 9$ also the condition $2 \leq b \leq \frac{n-2}{2}$ is fulfilled. Let us denote the first bound by $\underline{f_1}(n)$ and the second bound by $\overline{f_1}(n)$. In the following table we compare these bounds with the exact value $c_{\mathcal{C}}(n)$, determined using the methods from Section 5, and $\frac{\sqrt{n}}{2}$.

n	9	10	11	12	13	14	15	16
$\underline{f_1}(n)$	1.2727	1.3333	1.3846	1.4286	1.4667	1.5000	1.7143	1.7727
$\overline{f_1}(n)$	1.2000	1.3125	1.4118	1.5000	1.5789	1.6500	1.6296	1.7143
$c_{\mathcal{C}}(n)$	1.3333	1.4074	1.4667	1.5556	1.6500	1.7344	1.8088	1.8750
$\frac{\sqrt{n}}{2}$	1.5000	1.5811	1.6583	1.7320	1.8028	1.8708	1.9365	2.0000

In case 2 we obtain the optimization problem

$$\max \frac{n_1 n_2 - ab + b - a^2 + 2a + an_1 - 1 - n_1}{-a^2 - 2ab + a + an_1 + n_1 b + an_2 + b} \quad \text{s.t.}$$
$$a + b - 1 \geq n_1 + 1$$
$$n_1 + n_2 = n$$
$$n_1, n_2 \geq 1$$
$$1 \leq a \leq n_1 - 1$$
$$2 \leq b \leq n_2 - 2,$$

For $a > 1$ we can check that decreasing a, n_1 and increasing b, n_2 by 1 does not decrease the target value. Thus we can assume $a = 1$ in the optimal solution so that the target function simplifies to $\frac{n_1 n_2}{n_1(b+1)+(n_2-b)} = \frac{n_2}{b+1+\frac{n_2-b}{n_1}}$. Decreasing b by 1 increases this target function so that either $a + b - 1 \geq n_1 + 1$ or $b \geq 2$ is tight. In the latter case we would have $n_1 \leq 1$, which contradicts $1 = a \leq n_1 - 1$. Thus, we have $a + b - 1 = n_1 + 1$ in the optimum which is equivalent to $b = n_1 + 1$. Inserting this and $n_2 = n - n_1$ yields the target function

$$\frac{n - b + 1}{b + 1 + \frac{n - 2b + 1}{b - 1}}$$

having the non-negative optimal solution of $b = \frac{1 + \sqrt{1 + n^3 - 2n}}{n}$ with target value

$$\frac{1}{2} \cdot \frac{\sqrt{n^3 + 1 - 2n} - (n-1)}{n-1} \leq \frac{\sqrt{n}}{2}$$

tending to $\frac{\sqrt{n}}{2}$ as n approaches infinity. If the other inequalities are fulfilled, then rounding up or down yields the optimal integral solution (in this case; not in general). In both cases the conditions $2 \leq b \leq n_2 - 2$, $1 = a \leq n_1 - 1$ are fulfilled for $n \geq 9$. We produce a similar table as before:

n	9	10	11	12	13	14	15	16
$\underline{f_2}(n)$	1.1667	1.2308	1.2857	1.3333	1.3750	1.4118	1.4444	1.6250
$\overline{f_2}(n)$	1.0588	1.1667	1.2632	1.3500	1.4286	1.5000	1.5652	1.5484
$c_{\mathcal{C}}(n)$	1.3333	1.4074	1.4667	1.5556	1.6500	1.7344	1.8088	1.8750

Conjecture 2.

$$c_{\mathcal{C}}(n) \in \Theta\left(\sqrt{n}\right).$$

So far we do not know any examples of complete simple games with a critical threshold value larger than $\max\left(1, \frac{\sqrt{n}}{2}\right)$. We will prove Conjecture 2 for some special classes of complete simple games. An important class, used by many real-world voting systems, is given by the so-called games with consensus, i.e. intersections of a weighted voting game and a symmetric game $[q'; 1, \ldots, 1]$, see e.g. [Carreras and Freixas, 2004, Peleg, 1992]. The voting procedure for the council of the European Union based on the Treaty of Nice consists of such a consensus, i.e. at least 14 (or 18, if the proposal was not made by the commission) of the countries must agree. (The two other ingredients are a majority of the voting weights and a majority of the population.) Concerning the distribution of power in the European Union we refer the interested reader to e.g. [Algaba et al., 2007].

Lemma 11. *The critical threshold value $\mu(\chi)$ of a complete simple game $\chi \in \mathcal{C}_n$ with consensus, given as the intersection of $[q; w_1, \ldots, w_n]$ and $[q'; 1, \ldots, 1]$, is at most $\sqrt{n}$.*

Proof. If $q' \geq \sqrt{n}$ we take weights of $\frac{1}{\sqrt{n}}$ for all voters so that each winning coalition has a weight of at least one and the grand coalition a weight of $\sqrt{n}$. In the other cases we take weights $\frac{w_i}{q}$ for the voters so that each winning coalition has a weight of at least 1. W.l.o.g. we assume $w_i \leq q$ so that the new weights are at most 1. A losing

coalition with weight larger than one must fail the criterion of the symmetric game so that it consists of less than $\sqrt{n}$ members. Thus the weight of each losing coalition is less than $\sqrt{n}$.   □

For large consensus q' the critical threshold value is bounded from above by $\frac{n}{q'}$, since we can assign weights of $\frac{1}{q'}$ to all voters. We remark that complete simple games $((n_1, n_2), (m_1, m_2))$ with two equivalence classes of voters and one shift-minimal winning vector are games with consensus and thus have a dimension of at most two[3]. As representation we may use the intersection of $[m_1 + m_2; 1, \ldots, 1]$ and $[m_1 n_2 + m_2; n_2, \ldots, n_2, 1, \ldots, 1]$.

Lemma 12. *The critical threshold value $\mu(\chi)$ of a complete simple game $\chi \in \mathcal{C}_n$ with two types of voters is at most $\sqrt{n} + 1$.*

Proof. If χ has only one shift-minimal winning vector we can apply Lemma 11. Since complete simple games with less than four voters are weighted we can assume $n \geq 4$. So let $m_1 = (a, b)$ the shift-minimal winning vector with maximal a and $m_2 = (c, d)$ the shift-minimal winning vector with minimal c. Depending on the values of a and c we will provide suitable weights w_1 and w_2 such that each winning coalition has a weight of at least $q > 0$ and each losing coalition has a weight of at most $q \cdot (\sqrt{n} + 1)$, i.e. the proposed weights have to be normalized in order to fit into the framework of a quota $q = 1$.

If $c \geq 1$ we set $w_1 = \sqrt{n}$ and $w_2 = 1$. Every shift-minimal winning vector $(e, f) \neq (a, b)$ must fulfill $c \leq e \leq a$ due to the definition of a, c and $e + f \geq a + b + 1$ since otherwise (a, b) would not be a shift-minimal winning vector. With this we have

$$ew_1 + fw_2 \geq ew_1 + (a + b + 1 - e)w_2 \geq c\sqrt{n} + (a + b + 1 - c).$$

Similarly, we obtain

$$aw_1 + bw_2 = c\sqrt{n} + a - c + b + \underbrace{(a - c)}_{\geq 1} \cdot \underbrace{(\sqrt{n} - 1)}_{\geq 1} \geq c\sqrt{n} + (a + b + 1 - c).$$

Thus it suffices to show that each losing coalition has a weight of at most

$$\left(c\sqrt{n} + (a + b + 1 - c)\right) \cdot \left(\sqrt{n} + 1\right) \geq n + a\sqrt{n} + b\sqrt{n}.$$

Let (g, h) be a losing coalition so that $(g, h) \not\succeq (a, b)$ and $(g, h) \not\succeq (c, d)$. If $g \leq c$ then $h \leq n_2 \leq n - a$ and we have

$$gw_1 + hw_2 \leq c\sqrt{n} + n - a \leq n + a\sqrt{n}.$$

If $g \geq a$ then $g + h \leq a + b - 1$ since otherwise $(g, h) \succeq (a, b)$. With this we have

$$gw_1 + hw_2 \leq (a + b - 1)\sqrt{n} \leq a\sqrt{n} + b\sqrt{n}.$$

If $c \leq g < a$ then $g + h \leq c + d - 1$ since otherwise $(g, h) \succeq (c, d)$. With this we have

$$gw_1 + hw_2 \leq (a - 1)\sqrt{n} + (c + d - a) \leq n + a\sqrt{n}.$$

If $c = 0$ we set $w_1 = \sqrt{d}$, where $d \geq a + b + 1 \geq 2$, and $w_2 = 1$. Let (e, f) be a winning and (g, h) be a losing coalition. Similarly, as before we have $e + f \geq a + b$ so that

$$ew_1 + fw_2 \geq \sqrt{d} + a + b - 1.$$

It suffices to show that each losing coalition has a weight of at most

$$\left(\sqrt{d} + a + b - 1\right) \cdot \left(\sqrt{n} + 1\right) \geq \underbrace{\sqrt{dn} - \sqrt{n} + \sqrt{d}}_{\geq d} + (a + b)\sqrt{n} + \underbrace{a + b - 1}_{\geq 0}$$

$$\geq d + (a + b)\sqrt{n}.$$

If $g \geq a$ then $g + h \leq a + b - 1$, since otherwise $(g, h) \succeq (a, b)$, and we have

$$gw_1 + hw_2 \leq (a + b - 1)\sqrt{d} \leq (a + b)\sqrt{n}.$$

If $c \leq g < a$ then $g + h \leq c + d - 1$, since otherwise $(g, h) \succeq (c, d)$, and we have

$$gw_1 + hw_2 \leq (a - 1)\sqrt{d} + (c + d - a) \leq a\sqrt{n} + d.$$

                                                                                    □

---

[3]Complete simple games with one shift-minimal winning vector and more than two equivalence classes of voters can have dimensions larger than two and as large as $\frac{n}{4}$ [Freixas and Puente, 2008].

We remark that complete simple games with one type of voters are weighted and thus have a critical threshold value of 1.

Lemma 13. *The critical threshold value $\mu(\chi)$ of a complete simple game $\chi \in \mathcal{C}_n$ with one shift-minimal winning vector $\widetilde{a}$ is at most $\sqrt{n}$.*

Proof. By $(n_1, \ldots, n_t)$ we denote the numbers of voters in the $t \geq 2$ equivalence classes of voters and by $(a_1, \ldots, a_t)$ the unique shift-minimal winning vector $\widetilde{a}$.

If $\sum\limits_{i=1}^{t} a_i \geq \sqrt{n}$ we set $w_i = \frac{1}{\sqrt{n}}$ for all $1 \leq i \leq t$ and have $w(\widetilde{a}) \geq 1$. Since with these weights we have $w(N) \leq \sqrt{n}$, every losing coalition has a weight of at most $\sqrt{n}$ and we have a critical threshold value of at most $\sqrt{n}$.

In the remaining cases we have $\sum\limits_{i=1}^{t} a_i \leq \sqrt{n}$. Due to condition (a)(iii) of Theorem 1 we have $a_1 \geq 1$. We set $w_1 = 1$ and $w_2 = \cdots = w_t = 0$ and have $w(\widetilde{a}) \geq 1$. For every losing vector $\widetilde{l} = (l_1, \ldots, l_t)$ we have $l_1 < \sum\limits_{i=1}^{t} a_i \leq \sqrt{n}$ since otherwise we would have $\widetilde{a} \prec \widetilde{l}$. Thus each losing coalition has a weight of at most $\sqrt{n}$ and the critical threshold value is bounded from above by $\sqrt{n}$ in this case. $\qquad\square$

So, we have an upper bound of $\sqrt{n}$ for the critical threshold value for complete simple games on n voters in several subcases. For the general case of Conjecture 2 we can provide only a first preliminary bound showing that $c_{\mathcal{C}}(n)$ asymptotically grows slower than $c_{\mathcal{B}}(n)$ so that the maximum critical threshold value in some sense states that complete simple games are *nearer* to (roughly) weighted voting games than simple games.

Theorem 2. *The critical threshold value $\mu(\chi)$ of a complete simple game $\chi \in \mathcal{C}_n$ is in $O\!\left(\frac{n \cdot \log\log n}{\log n}\right)$.*

Proof. As weights we choose a slowly decreasing geometric series $w_i = q^{i-1}$ for all $1 \leq i \leq n$ where $q = 1 - \frac{\log n}{n \cdot \log\log n}$. With this we have $0 \leq q < 1$ and $\frac{1}{1-q} = \frac{n \cdot \log\log n}{\log n}$. Now, let W be a winning coalition with the minimum weight and L be a losing coalition with the maximum weight. In the following we will show $\frac{w(L)}{w(W)} \leq \frac{n \cdot \log\log n}{\log n}$. To deduce this bound we will compare the weights of a few subsets of consecutive voters. In order to keep the necessary number of such subsets small, we set $\widetilde{W} := W \backslash (W \cap L)$ and $\widetilde{L} := L \backslash (W \cap L)$, i.e. we technically remove common voters. We remark that $\widetilde{W}$ needs not be a winning coalition. Due to the inequality

$$\frac{x}{y} \geq \frac{x+c}{y+c}$$

for $x \geq y > 0$ and $c \geq 0$ it suffices to provide an upper bound for $\frac{w(\widetilde{L})}{w(\widetilde{W})}$.

At first, we consider the case when W is lexicographically larger than L. Let j be the voter with the minimal index (and so the maximal weight) in $\widetilde{W}$. With this we set $W' = \{j\}$, $L' = \{j+1, \ldots, n\}$ and have $w(\widetilde{W}) \geq w(W')$, $w(\widetilde{L}) \leq w(L')$ so that $\frac{w(\widetilde{L})}{w(\widetilde{W})}$ is upper bounded by

$$\frac{w(\widetilde{L})}{w(\widetilde{W})} \leq \frac{w(L')}{w(W')} = \frac{q(1 - q^{n-j})}{1 - q} \leq \frac{1}{1-q} = \frac{n \cdot \log\log n}{\log n}.$$

If W is lexicographically smaller than L then let j be an index with $\underbrace{|\widetilde{W} \cap \{1, \ldots, j\}|}_{=:k_1} > \underbrace{|\widetilde{L} \cap \{1, \ldots, j\}|}_{=:k_2}$. With this we set $L' := \{1, \ldots, k_2\} \cup \{j+1, \ldots, n\}$ and $W' := \{j - k_1 + 1, \ldots, j\}$ fulfilling $w(\widetilde{W}) \geq w(W')$ and $w(\widetilde{L}) \leq w(L')$. Since $k_1 \geq k_2 \geq 1$,

$$w(L') = \sum_{i=1}^{k_2} q^{i-1} + \sum_{i=j+1}^{n} q^{i-1} = \frac{1 - q^{k_2}}{1 - q} + q^j \cdot \frac{1 - q^{n-j}}{1 - q}$$

and $w(W') = q^{j-k_1+1} \cdot \frac{1 - q^{k_1}}{1-q} \geq q^{j-k_1+1}$ we have

$$\frac{w(L)}{w(W)} \leq \frac{w(\widetilde{L})}{w(\widetilde{W})} \leq \frac{w(L')}{w(W')} \leq q^{k_1 - j - 1} + \frac{q^j \cdot \frac{1}{1-q}}{q^{j-k_1+1}} \leq q^{-j} + \frac{1}{1-q} \leq q^{-n} + \frac{1}{1-q}.$$

To finish the proof we show $q^{-n} \in O\left(\frac{n \cdot \log \log n}{\log n}\right)$. From $\frac{x}{1+x} \le \log(1+x) \le x$ for $x > -1$ we conclude $2x \ge \frac{x}{1-x} \ge -\log(1-x) \ge x$ for $\frac{1}{2} \le x \le 1$. Thus for large enough n we have

$$\log\left(q^{-n}\right) \le n \cdot \left(-\log\left(1 - \frac{\log n}{n \cdot \log \log n}\right)\right) \le n \cdot \frac{2\log n}{n \cdot \log \log n} \le \frac{2 \log n}{\log \log n}$$

and $\frac{2 \log n}{\log \log n} \le \log n - \log \log n + \log \log \log n = \log\left(\frac{n \cdot \log \log n}{\log n}\right).$ $\square$

In the context of the conjectured upper bound of $O(\sqrt{n})$ for $c_{\mathcal{C}}(n)$ we find it remarkable that the cost of stability $CoS(f)$ of any super-additive, see Definition 5, Boolean game $f \in \mathcal{B}_n$ is upper bounded by $\sqrt{n} - 1$, see [Bachrach et al., 2009]. If f is additionally anonymous, then the authors have proven the tighter bound $CoS(f) \le 2$. This coincides with the situation for the critical threshold value. Here we may consider the super-additive anonymous Boolean game $f \in \mathcal{B}_n$, where coalitions of size $\lceil \frac{n+1}{2} \rceil$ are winning and the grand coalition N is losing.

5. AN INTEGER LINEAR PROGRAMMING APPROACH TO DETERMINE THE MAXIMAL CRITICAL THRESHOLD VALUE

In principle it is possible to determine the maximal critical threshold value $c_{\mathcal{S}}(n)$ for a given integer n by simply solving the stated linear program from Proposition 1 for all simple games $\chi \in \mathcal{S}_n$. Since for $n \le 8$ there are 1, 4, 18, 166, 7 579, 7 828 352, 2 414 682 040 996, and 56 130 437 228 687 557 907 786 simple games, an exhaustive search seems to be hopeless even for moderate n (of course theoretical results may help to reduce the number of simple games which need to be checked). For $n = 9$ only the lower bound 10^{42} is known.

So, alternatively we will formulate $c_{\mathcal{S}}(n)$ as the solution of an optimization problem in the following to avoid exhaustive enumeration. It is possible to describe the set of monotone Boolean functions as integer points of a polyhedron, see e.g. [Kurz, 2012b]: For each subset $S \subseteq N$ we introduce a binary variable x_S and use the constraints $x_\emptyset = 0$, $x_N = 1$, and $x_{S\setminus\{i\}} \le x_S$ for all $\emptyset \ne S \subseteq N$, $i \in S$ to model a simple game via $\chi(S) = x_S$. (We have to remark that this ILP formulation is very *symmetric*.) In this framework it is very easy to add additional restrictions. Methods to restrict the underlying games to complete simple games or weighted voting games are outlined in [Kurz, 2012b]. The restriction to e.g. *proper* simple games can be modeled via $x_S + x_{N\setminus S} \le 1$ for all $S \subseteq N$. Similarly, *strong* simple games can be modeled by using the constraints $x_S + x_{N\setminus S} \ge 1$ for all $S \subseteq N$.

So the problem of determining $c_{\mathcal{S}}(n)$ can be stated as the following optimization problem: Maximize over all simple games with n voters the minimum α of the linear program (1). Since this is a two-level optimization problem, we have to reformulate the problem in order to apply integer linear programing techniques.

In order to determine $c_{\mathcal{S}}(n)$ we cannot maximize α directly since we have $\chi \in \mathcal{T}_{\lambda\alpha} \cap \mathcal{S}_n$ for all $\lambda \ge 1$ if $\chi \in \mathcal{T}_\alpha \cap \mathcal{S}_n$. To specify the minimum value α for a given simple game χ we can also maximize its corresponding dual linear program of (1) whose optimal solution is α.

If we drop the restriction $\alpha \ge 1$ and assume $w_i \ge 0$, the dual program for a simple game χ is given by

$$
\begin{aligned}
\text{Max} \quad & \sum_{S \in W} u_S \\
& \sum_{S \in W: i \in S} u_S - \sum_{S \in L: i \in S} v_S \le 0 \quad \forall i \in N \\
& \sum_{S \in L} v_S \le 1 \\
& u_S \ge 0 \quad \forall S \in W \\
& v_S \ge 0 \quad \forall S \in L,
\end{aligned}
$$

where W denotes the set of winning coalitions and L denotes the set of losing coalitions. As outlined in Section 2 the optimal target value $\sum_{S \in W} u_S$ might take values smaller than 1 (but being non-negative) which correspond to a critical threshold value of $\mu(\chi) = 1$.

The next step is to replace the externally given sets W and L by variables such that the possible sets correspond to simple games. Using our previously defined binary variables x_S this is rather easy:

$$
\begin{aligned}
\text{Max} \quad & \sum_{S \subseteq N} x_S \cdot u_S \\
& \sum_{\{i\} \subseteq S \subseteq N} x_S \cdot u_S - \sum_{\{i\} \subseteq S \subseteq N} (1 - x_S) \cdot v_S \leq 0 \quad \forall i \in N \\
& \sum_{S \subseteq N} (1 - x_S) \cdot v_S \leq 1 \\
& x_\emptyset = 0 \\
& x_N = 1 \\
& x_{S \setminus \{i\}} \leq x_S \quad \forall \emptyset \neq S \subseteq N \\
& u_S \geq 0 \quad\quad\ \forall S \subseteq N \\
& v_S \geq 0 \quad\quad\ \forall S \subseteq N \\
& x_S \in \{0, 1\} \quad \forall S \subseteq N,
\end{aligned}
$$

The problem is a quadratically constrained quadratic program (QCQP) with binary variables or more generally a mixed-integer quadratically constrained program (MIQCP). There are solvers, like e.g. `ILOG CPLEX`, that can deal with these problems efficiently whenever the target function and the constraints are convex. Unfortunately, neither our target function nor the feasibility set is convex. Thus in order to solve this optimization problem directly, we have to utilize a solver that can deal with non-convex mixed-integer quadratically constrained programs like e.g. `SCIP`, see e.g. [Berthold et al., 2011a, Berthold et al., 2011b][4].

This works in principle, but problems become computationally infeasible very quickly. By disabling preprocessing we can force `SCIP` to use general MIQCP-techniques. Solving the problem Boolean functions with $f(\emptyset) = 0$ and $n = 3$ took 0.07 seconds and 43 b&b-nodes, for $n = 4$ it took 8.45 seconds and 15770 b&b-nodes, and for $n = 5$ we have aborted the solution process after 265 minutes and $1.6 \cdot 10^6$ nodes, where more than 33 GB of memory was used.

By enabling preprocessing `SCIP` is able to automatically find a reformulation as a binary linear program. This way `SCIP` can solve the instance for $n = 8$ in 2.9 seconds in the root node but will take more than 211 minutes, 373000 nodes, and 1.8 GB of memory to solve the instance for $n = 9$.

Since often binary linear programs are easier to solve than binary quadratic problems, we want to reformulate our binary quadratic optimization problem into a binary linear one. There are several papers dealing with reformulations of MIQCPs into easier problems, see e.g. [Letchford and Galli, 2011]. Here we want to present a custom-tailored approach based on some techniques that are quite standard in the mixed integer linear programing community (but we will outline them nevertheless). Using this formulation, `SCIP` needed only 18.72 seconds to solve the instance for $n = 15$ without applying branch&bound. We would like to remark that `CPLEX` was even faster using only 5.61 seconds of computation time.

A quite general technique to get rid of logical implications are so called Big-M constraints, see e.g. [Koch, 2004]. To explain the underlying concept we consider a binary variable $x \in \{0, 1\}$, a real-valued variable y, and a *conditional* inequality $y \leq c$ for a constant c, which only needs to be satisfied if $x = 1$. The idea is to use this inequality, but to modify its right-hand side with a constant times $(1 - x)$:

$$
y \leq c + (1 - x) \cdot M.
$$

For $x = 1$ this inequality is equivalent to the desired *conditional* inequality. Otherwise the new inequality is equivalent to $y \leq c + M$, which is satisfied if M is large enough. Given a known upper bound $y \leq u$, where possibly $u \gg c$, it suffices to choose $M = u - c$.

Now we want to apply this technique in a more sophisticated way, to remove the non-linear term $x_S \cdot u_S$, where $x_S \in \{0, 1\}$ and $u_S \in [0, \beta]$. We replace the term $x_S \cdot u_S$ by the variable $z \geq 0$ using the constraints $z \leq \beta x_S$, $z \leq u_S$, and $z \geq u_S - \beta (1 - x_S)$. If $x_S = 1$ these inequalities state that $z = x_S \cdot u_S = u_S$ must hold and for $x_S = 0$ they imply $z = x_S \cdot u_S = 0$. Thus one extra variable and three additional inequalities are necessary for each term of the form $x_S \cdot u_S$ or $x_S \cdot v_S$. The LP relaxation gets worser with increasing β, the so-called Big-M constant. Of course in general, it may be hard to come up with a concrete bound β. In our case it is not too hard to prove $u_S, v_S \leq 1$: If $x_T = 0$ then from $v_S \geq 0$ for all $S \subseteq N$ and $\sum_{S \subseteq N} (1 - x_S) \cdot v_S \leq 1$ we conclude $v_T \leq 1$.

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[4]We have to remark that currently SCIP is not capable of solving the stated problem without further information because there are some problems if the intermediate LP relaxations are unbounded. So one has to provide upper and lower bounds for the continuous variables u_S and v_S.

Otherwise v_T does not occur anywhere in the optimization problem and $v_T \leq 1$ is a valid inequality. Similarly, if $x_T = 0$ then u_T does not appear anywhere and on the other hand for $x_T = 1$ we have

$$u_T \leq \sum_{\{i\} \subseteq S \subseteq N} (1 - x_S) \cdot v_S \leq \sum_{S \subseteq N} (1 - x_S) \cdot v_S \leq 1.$$

Due to the special structure of our problem we can reformulate our problem without additional variables and fewer additional constraints. The main idea is to use the term u_S instead of $x_S \cdot u_S$ and to ensure that we have $u_S = 0$ for $x_S = 0$. Similarly, we replace the products $(1 - x_S) \cdot v_S$ by v_S and ensure that we have $v_S = 0$ if $x_S = 1$.

$$(17) \qquad\qquad \max \qquad \sum_{S \subseteq N} u_S$$

$$(18) \qquad\qquad\qquad\qquad x_\emptyset \;=\; 0$$

$$(19) \qquad\qquad\qquad\qquad x_N \;=\; 1$$

$$(20) \qquad\qquad x_S - x_{S \setminus \{i\}} \;\geq\; 0 \quad \forall \emptyset \neq S \subseteq N, i \in S$$

$$(21) \qquad \sum_{\{i\} \subseteq S \subseteq N} u_S - \sum_{\{i\} \subseteq S \subseteq N} v_S \;\leq\; 0 \quad \forall i \in N$$

$$(22) \qquad\qquad\qquad \sum_{S \subseteq N} v_S \;\leq\; 1$$

$$(23) \qquad\qquad\qquad\qquad u_S \;\leq\; x_S \quad \forall S \subseteq N$$

$$(24) \qquad\qquad\qquad\qquad v_S \;\leq\; 1 - x_S \quad \forall S \subseteq N$$

$$(25) \qquad\qquad\qquad\qquad x_S \;\in\; \{0, 1\} \quad \forall S \subseteq N$$

$$(26) \qquad\qquad\qquad\qquad u_S \;\geq\; 0 \quad \forall S \subseteq N$$

$$(27) \qquad\qquad\qquad\qquad v_S \;\geq\; 0 \quad \forall S \subseteq N$$

Inequalities (21) and (22) capture the dual linear program to bound $\alpha = \sum_{S \subseteq N} u_S$ from above. Inequality (23) models the implication that u_T is zero if $x_T = 0$. In the other case where $x_T = 1$ the inequality $u_T \leq 1$ is redundant since we have for an $i \in T$ ($x_\emptyset = 0$) the inequality $\sum_{\{i\} \subseteq S \subseteq N} u_S - \sum_{\{i\} \subseteq S \subseteq N} \leq 0$ from which we conclude $x_T \leq 1$ using $x_S \geq 0$ and $\sum_{S \subseteq N} v_S \leq 1$. Inequalities of that type are called Big-M inequalities, where we have an *Big-M* of 1 in our two cases. (See Inequality (34) for an example with a *Big-M* constant larger than 1.) Similarly, Inequality (24) models the implication that v_T is zero if $x_T = 1$. In the other case where $x_T = 0$ we have the redundant inequality $v_T \leq 1$.

The optimum target value of this ILP is the desired value $c_S(n)$ for each integer n. We have to remark that our modeling of the set of simple games is highly symmetric and each solution comes with at least $n!$ isomorphic solutions which is an undesirable feature for an ILP model. With the stated ILP model we were able to computationally prove Conjecture 1 for $n \leq 9$ taking less than 37 seconds for $n = 7$, less than 279 seconds for $n = 8$ but already 66224 seconds and 161898779 branch&bound nodes for $n = 9$. For $n = 10$ we have computationally obtained the bounds $\frac{5}{2} \leq c_S(10) \leq 3$ from an aborted ILP solution process. (The LP relaxation gives only the relatively poor upper bound of $\frac{n-1}{2}$.)

We would like to remark that we can enhance this ILP formulation a bit. Since we have $c_S(n+1) \geq c_S(n)$ we may apply Lemma 10 and require $x_{\{i\}} = 0$ for all $1 \leq i \leq n$, where $n \geq 2$.

If we replace conditions (20) by those for complete simple games we can determine the exact values $c_C(n)$ for $n \leq 16$: $c_C(1) = c_C(2) = c_C(3) = c_C(4) = c_C(5) = c_C(6) = 1$, $c_C(7) = \frac{8}{7}$, $c_C(8) = \frac{26}{21}$, $c_C(9) = \frac{4}{3}$, $c_C(10) = \frac{38}{27}$, $c_C(11) = \frac{22}{15}$, $c_C(12) = \frac{14}{9}$, $c_C(13) = \frac{33}{20}$, $c_C(14) = \frac{111}{64}$, $c_C(15) = \frac{123}{68}$, and $c_C(16) = \frac{15}{8}$.

We would like to remark that the LP relaxation gives only the poor upper bound $c_C(n) \leq \frac{n-1}{2}$.

6. THE SPECTRUM OF CRITICAL THRESHOLD VALUES

In sections 4 and 5 we have considered the maximum critical threshold value for several classes of games. By $Spec_S(n)$ we denote the entire set of possible critical threshold values of simple games on n voters. Similarly, we define $Spec_B(n)$ as the set of possible critical threshold values for Boolean functions $f : 2^N \to \{0, 1\}$ with

$f(\emptyset) = 0$ and $Spec_{\mathcal{C}}(n)$ as the set of possible critical threshold values for complete simple games on n voters. In this section we will provide a superset for the spectrum using known information of the set of possible determinants of 0/1 matrices. In order to compute the exact sets for small values of n we modify the presented integer linear programming approach for the determination of the maximum critical threshold value to that end.

By considering null voters we conclude $Spec_{\mathcal{S}}(n) \subseteq Spec_{\mathcal{S}}(n+1)$, $Spec_{\mathcal{B}}(n) \subseteq Spec_{\mathcal{B}}(n+1)$, and $Spec_{\mathcal{C}}(n) \subseteq Spec_{\mathcal{C}}(n+1)$. Due to the inclusion of the classes of games we obviously have $Spec_{\mathcal{C}}(n) \subseteq Spec_{\mathcal{S}}(n) \subseteq Spec_{\mathcal{B}}(n)$ for all $n \in \mathbb{N}$.

Principally, it is possible to determine the sets $Spec_{\mathcal{S}}(n)$ for small numbers of voters by exhaustive enumeration of all simple games. As mentioned in the previous section this approach is very limited due to the quickly increasing number of simple games. In [Gvozdeva et al., 2012] the authors have determined $Spec_{\mathcal{S}}(n)$ for all $n \leq 6$ by some theoretical reductions and exhaustive enumeration on the restricted set of possible games.

In this section we want to develop an approach based on integer linear programming to determine the spectrum and to utilize results on Hadamard's maximum determinant problem to obtain a superset of the spectrum. For the latter let us consider the linear program (1) determining the critical threshold value of a Boolean function with $f(\emptyset) = 0$. Each element of the spectrum $Spec_{\mathcal{B}}(n)$ appears as the optimal solution of this linear program for a certain Boolean function f. If inequality $\alpha \geq 1$ is attained with equality in the optimal solution, the critical threshold value is 1. So we may drop this inequality and consider only those functions f where the linear program (1) without the inequality $\alpha \geq 1$ has an optimal solution, which is then attained in a corner. Thus there are subsets $W_1, \ldots, W_k \subseteq N$, where $0 \leq k \leq n+1$, with $\sum_{j \in W_i} w_j = 1$ and $n+1-k$ subsets $L_1, \ldots, L_{n+1-k} \subseteq N$ with $-\alpha + \sum_{j \in L_i} w_j = 0$ such that the entire linear equation system has a unique solution. (We remark that $k = 0$ and $k = n+1$ lead to infeasible solutions.)

Writing this equation system in matrix notation $A \cdot (w_1, \ldots, w_n, \alpha)^T = b$ we can use Cramer's rule to state

$$\alpha = \frac{\det(A_\alpha)}{\det(A)},$$

where A_α arises from A by replacing the rightmost column by b. Since A_α is a 0/1-matrix we can use an improved version of Hadamard's bound and have

$$|\det(A_\alpha)| \leq \frac{(n+2)^{(n+2)/2}}{2^{n+1}},$$

see e.g. [Brenner and Cummings, 1972]. If we multiply the rightmost column of A by -1, which changes the determinant by a factor of $(-1)^{n+1}$ then it becomes a 0/1-matrix too and we conclude

$$|\det(A)| \leq \frac{(n+2)^{(n+2)/2}}{2^{n+1}}.$$

Lemma 14. *For each $\alpha \in Spec_{\mathcal{B}}(n)$ there are coprime integers $1 \leq q < p \leq \left\lfloor \frac{(n+2)^{(n+2)/2}}{2^{n+1}} \right\rfloor$ with $\alpha = \frac{p}{q}$.*

For specific n the uppers bounds on the determinant of 0/1-matrices can be improved. The exact values for the maximum determinant of a $n \times n$ binary matrix for $n \leq 17$ are given by $1, 1, 2, 3, 5, 9, 32, 56, 144, 320, 1458, 3645, 9477, 25515, 131072, 327680, 1114112$, see e.g. sequence A003432 in the on-line encyclopedia of integer sequences and the references therein.

Another restriction on the possible critical threshold values is obviously given by the maximum values, i.e. $\mu(\chi) \leq c_{\mathcal{B}}(n)$ (or $\mu(\chi) \leq c_{\mathcal{S}}(n)$ for simple games, $\mu(\chi) \leq c_{\mathcal{C}}(n)$ for complete simple games. Further restrictions come from the possible spectrum of determinants of binary matrices. For binary $n \times n$-matrices all determinants between zero and the maximal value can be attained. For $n \geq 7$ gaps occur, see e.g. [Craigen, 1990]. The spectrum of the determinants of binary 7×7-matrices was determined in [Metropolis, 1971] to be $\{1, \ldots, 18\} \cup \{20, 24, 32\}$. Using this more detailed information we can conclude that the denominator q of the critical threshold value of a Boolean function with $f(\emptyset) = 0$ on 6 voters is at most 17. Thus, we are able to compute a finite superset $\Lambda(n)$ of $Spec_{\mathcal{B}}(n)$ for each number n of voters.

Our next aim is to provide an ILP formulation in order to determine the entire spectrum for simple games and complete simple games on n voters. Therefore, we consider the linear program (1) for the determination of the critical threshold value. Dropping the constraint $\alpha \geq 1$ and assuming $w_i \geq 0$ we abbreviate the emerging linear program by $\min c^T x$, $Ax \geq b, x \geq 0$. If its optimal value is at least 1 then it coincides with the critical threshold value. Otherwise the game is weighted. By the strong duality theorem its dual $\max b^T y$ $A^T y \leq c, y \geq 0$ has the same optimal solution if both are feasible. This is indeed the case taking the dual solution $y = 0$ and primal weights

of 1 with an α of n. Thus, we can read of the critical threshold value as $c^T x$ from each feasible solution of the inequality system $Ax \geq b$, $A^T y \leq c$, $c^T x = b^T y$, $x, y \geq 0$.

As done in Section 5 we model the underlying simple game by binary variables x_S for the subsets $S \subseteq N$ and use Big-M constraints:

$$(28) \qquad x_\emptyset = 0$$

$$(29) \qquad x_N = 1$$

$$(30) \qquad x_S - x_{S \setminus \{i\}} \geq 0 \quad \forall \emptyset \neq S \subseteq N, i \in S$$

$$(31) \qquad x_S \in \{0, 1\} \quad \forall S \subseteq N$$

$$(32) \qquad w_i \leq 1 \quad \forall i \in N$$

$$(33) \qquad \sum_{i \in S} w_i \geq x_S \quad \forall S \subseteq N$$

$$(34) \qquad \sum_{i \in S} w_i \leq \alpha + |S| \cdot x_S \quad \forall S \subseteq N$$

$$(35) \qquad w_n \geq 0$$

$$(36) \qquad \sum_{\{i\} \subseteq S \subseteq N} u_S - \sum_{\{i\} \subseteq S \subseteq N} v_S \leq 0 \quad \forall i \in N$$

$$(37) \qquad \sum_{S \subseteq N} v_S \leq 1$$

$$(38) \qquad u_S \leq x_S \quad \forall S \subseteq N$$

$$(39) \qquad v_S \leq 1 - x_S \forall S \subseteq N$$

$$(40) \qquad u_S \geq 0 \quad \forall S \subseteq N$$

$$(41) \qquad v_S \geq 0 \quad \forall S \subseteq N$$

$$(42) \qquad \sum_{S \subseteq N} u_S = \alpha$$

Inequalities (28)-(31) model the simple games. The primal program to determine the critical threshold value is given as inequalities (32)-(35). W.l.o.g. we can restrict the weights to lie inside $[0, 1]$. Inequality (33) states that the weight of each winning coalition is at least 1 and that the weight of each losing coalition is at least zero, which is a valid inequality. Similarly, Inequality (34) is fulfilled for $x_S = 1$ and translates to $w(S) \leq \alpha$ for each losing coalition S. The formerly used dual linear program is stated in inequalities (36)-(41). Finally the coupling of the primal and the dual target value is enforced in Inequality (42).

We remark that in order to destroy a bit of the inherent symmetry, i.e. the group of all permutations on n elements acts on the set of simple games, we might require $w_1 \geq \cdots \geq w_n$.

Having this inequality system at hand, one may prescribe each element in $\Lambda(n)$ as a possible value for α and check whether it is feasible, then α is contained in the spectrum, or not.

Another possibility to determine the entire spectrum is to solve a sequence of ILPs, where we add the target function $\min \alpha$ and the constraint $\alpha \geq l$. As starting value we choose $l = \min\{v \in \Lambda(n) : v > 1\}$. If the optimal target value is given by α', we choose $l = \min\{v \in \Lambda(n) : v > \alpha'\}$ until the set is empty. We remark that for larger n the values of $\Lambda(n)$ might be relatively close to each other so that numerical problems may occur.

Using the latter approach, we have verified the results $Spec_{\mathcal{S}}(1) = Spec_{\mathcal{S}}(2) = Spec_{\mathcal{S}}(3) = Spec_{\mathcal{S}}(4) = \{1\}$, $Spec_{\mathcal{S}}(5) = \left\{1, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}, \frac{9}{8}\right\}$, and $Spec_{\mathcal{S}}(6) = Spec_{\mathcal{S}}(5) \cup \left\{\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{9}{7}, \frac{10}{9}, \frac{11}{9}, \frac{11}{10}, \frac{12}{11}, \frac{13}{10}, \frac{13}{11}, \frac{13}{12}, \frac{14}{11}, \frac{14}{13}, \right.$ $\left. \frac{15}{13}, \frac{15}{14}, \frac{16}{13}, \frac{16}{15}, \frac{17}{13}, \frac{17}{14}, \frac{17}{15}, \frac{17}{16}\right\}$ already given in [Gvozdeva et al., 2012]. For $n = 7$ we have newly determined the smallest non-trivial critical threshold value $\min Spec_{\mathcal{S}}(7) \setminus \{1\} = \frac{40}{39}$. [5] For $n = 8$ we conjecture $\min Spec_{\mathcal{S}}(8) \setminus \{1\} = \frac{105}{104}$ [6].

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[5]Since the possible spectrum of determinants is given by $\{0, \ldots, 40, 42, 44, 45, 48, 56\}$, see e.g. http://www.indiana.edu/~maxdet/spectrum.html, only $\frac{45}{44}$ had to be ruled out.

[6]Here the possible spectrum of determinants is given by $\{0, \ldots, 102, 104, 105, 108, 110, 112, 116, 117, 120, 125, 128, 144\}$ so that only $\frac{117}{116}$ might be possible.

By dropping the inequalities (29), (30) and permitting negative weights, i.e. $w_i \in \mathbb{R}$, we can principally determine the entire spectrum for Boolean functions with $f(\emptyset) = 0$. For small n, the explicit sets are given by

$$
\begin{aligned}
Spec_{\mathcal{B}}(1) &= \{1\} \\
Spec_{\mathcal{B}}(2) &= \{1, 2\} \\
Spec_{\mathcal{B}}(3) &= \left\{1, \frac{3}{2}, 2, 3\right\} \\
Spec_{\mathcal{B}}(4) &= \left\{1, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, 2, \frac{5}{2}, 3, 4\right\} \\
Spec_{\mathcal{B}}(5) &= \left\{1, \frac{9}{8}, \frac{8}{7}, \frac{7}{6}, \frac{6}{5}, \frac{5}{4}, \frac{9}{7}, \frac{4}{3}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, 2, \frac{9}{4}, \frac{7}{3}, \frac{5}{2}, \frac{8}{3}, 3, \frac{7}{2}, 4, 5\right\} \\
Spec_{\mathcal{B}}(6) &= \left\{1, \frac{18}{17}, \frac{17}{16}, \frac{16}{15}, \frac{15}{14}, \frac{14}{13}, \frac{13}{12}, \frac{12}{11}, \frac{11}{10}, \frac{10}{9}, \frac{9}{8}, \frac{17}{15}, \frac{8}{7}, \frac{15}{13}, \frac{7}{6}, \frac{13}{11}, \frac{6}{5}, \frac{17}{14}, \frac{11}{9}, \right. \\
&\qquad \frac{16}{13}, \frac{5}{4}, \frac{14}{11}, \frac{9}{7}, \frac{13}{10}, \frac{17}{13}, \frac{4}{3}, \frac{15}{11}, \frac{11}{8}, \frac{18}{13}, \frac{7}{5}, \frac{17}{12}, \frac{10}{7}, \frac{13}{9}, \frac{16}{11}, \frac{3}{2}, \frac{17}{11}, \frac{14}{9}, \frac{11}{7}, \\
&\qquad \frac{8}{5}, \frac{13}{8}, \frac{18}{11}, \frac{5}{3}, \frac{17}{10}, \frac{12}{7}, \frac{7}{4}, \frac{16}{9}, \frac{9}{5}, \frac{11}{6}, \frac{13}{7}, \frac{15}{8}, \frac{17}{9}, 2, \frac{17}{8}, \frac{15}{7}, \frac{13}{6}, \frac{11}{5}, \frac{9}{4}, \\
&\qquad \left. \frac{16}{7}, \frac{7}{3}, \frac{12}{5}, \frac{5}{2}, \frac{13}{5}, \frac{8}{3}, \frac{11}{4}, \frac{14}{5}, 3, \frac{13}{4}, \frac{10}{3}, \frac{7}{2}, \frac{11}{3}, 4, \frac{9}{2}, 5, 6\right\}
\end{aligned}
$$

For complete simple games we simply replace the conditions (28)-(31) by those for complete simples games. As complete simple games with up to 6 voters are roughly weighted, we have $Spec_{\mathcal{C}}(n) = \{1\}$ for $n \leq 6$. For $n = 7$ we have determined $\min\left\{Spec_{\mathcal{C}}(7)\backslash\{1\}\right\} = \frac{39}{38}$.

7. CONCLUSION

In this paper we have considered the critical threshold values for several subclasses of binary voting structures. For Boolean games an exact upper bound of $\mu_{\mathcal{B}}(n) = n$ could be determined. The set of achievable values is strongly related to the spectrum of determinants of binary matrices, so that Hadamard's bound comes into play.

We have strengthened the lower and upper bound on the maximum critical threshold value of a simple game on n voters to $\left\lfloor \frac{n^2}{4} \right\rfloor / n \leq c_{\mathcal{S}}(n) \leq \frac{n}{3}$. It remains to prove (or to disprove) the conjecture that the lower bound is tight. By introducing an integer linear programming approach to determine the maximum critical threshold value we could algorithmically verify this conjecture for all $n \leq 9$. On the one hand, this seems to be a rather small number. On the other hand, regarding the question of the number of simple games, not much more than a lower bound of 10^{42} is known. Since the number of simple games grows doubly exponential, no huge improvements can be expected from an algorithmic point of view.

For complete simple games the problem to determine $c_{\mathcal{C}}(n)$ is considerably harder. The large gap between the stated upper bound $\frac{c_1 n \log \log n}{\log n}$ and lower bound $c_2 \sqrt{n}$ deserves to be closed or at least to be narrowed. In order to facilitate the conjectured asymptotics of $\Theta(\sqrt{n})$ we have provided a class of examples achieving this bound and have proven the respective upper bounds for several subclasses of complete simple games.

So far we have no structural insights on those complete simple games which achieve $c_{\mathcal{C}}(n)$ as their critical threshold value. The given integer linear programming formulation for $c_{\mathcal{C}}(n)$ made it possible to determine exact values for numbers of voters where even the number of complete simple games is not known. To be more precise, there are $284\,432\,730\,174$ complete simple games for nine voters, see e.g. [Kurz, 2012a] or [Freixas and Molinero, 2010], while exact numbers are unknown for $n \geq 10$. The fact that the exact numbers for the critical threshold values $c_{\mathcal{C}}(n)$ for complete simple games are known up to $n = 16$, indicates the great potential of our introduced algorithmic approach. Similar integer linear programming formulations can possibly be developed for other problems on extremal voting schemes. Applications to related concepts like, e.g., the nucleolus or the cost of stability seem to be promising.

In this paper we leave the question for the complexity to determine the criticial threshold value within a given class of games open, but expect it to be in NP in general.

Concerning the discriminability of the hierarchy of α-roughly weighted simple games, it would be nice to prove (if true) that there is a complete simple game χ with critical threshold value $\mu(\chi) = \frac{p}{q}$ for all integers $p \geq q$. Some first experiments let us conjecture that there even is a complete simple game with two types of voters and one shift-minimal winning vector.

As usual, the relation to other solution concepts from the game theory literature to the critical threshold value should be studied. We have started this task by considering the cost of stability. Is turns out that the critical threshold value is upper bounded by the cost of stability. From that, we could deduce an upper bound of $\sqrt{n}$ for super-additive games. For Boolean games the asymptotic extremal values coincide, while they can differ to a large extent for concrete games.

The maximum critical threshold value can discriminate between the classes of simple games, complete simple games, and weighted voting games, while the cost of stability can not. The concept of a dimension of a simple game is not directly related to the critical threshold value.

The concept of α-weightedness seems very interesting. More research should be done in that direction. A quite natural idea is to transfer the concept to ternary voting games, see e.g. [Felsenthal and Machover, 1997] and [Freixas and Zwicker, 2003], or graph based games like e.g. network flow games. Also effectivity functions, see e.g. [Storcken, 1997], might be candidates for a generalization of the basic concept. Last but not least, there are two additional hierarchies of simple games described in [Gvozdeva et al., 2012] which deserve to be analyzed in more detail.

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APPENDIX A. FURTHER SIDE RESULTS

In this appendix we mention some additional results, which are obtained with the techniques described in the paper, but are a bit to specific to be included in the main part.

A.1. **Strong or proper simple games.** In Section 5 we have mentioned that one can easily model restrictions within the class of simple games, e.g. consider proper or strong simple games. So, for each voting class $\mathcal{X} \in \{\mathcal{B}, \mathcal{S}, \mathcal{C}\}$ let $c_{\mathcal{X}}^s(n)$ denote the maximum critical threshold value of a game consisting of n voters in $\mathcal{X}$, which is strong. Similarly, we define $c_{\mathcal{X}}^p(n)$ for games which are proper and $c_{\mathcal{X}}^{ps}(n)$ for games which are proper and strong. Numerical results for small numbers of voters are stated in Table 1.

n	$c_{\mathcal{C}}(n)$	$c_{\mathcal{C}}^p(n)$	$c_{\mathcal{C}}^s(n)$	$c_{\mathcal{C}}^{ps}(n)$
7	$\frac{8}{7} \approx 1.142857$	$\frac{14}{13} \approx 1.076923$	$\frac{10}{9} = 1.\overline{1}$	1
8	$\frac{26}{21} \approx 1.238095$	$\frac{38}{33} = 1.\overline{15}$	$\frac{26}{21} \approx 1.238095$	1
9	$\frac{4}{3} = 1.\overline{3}$	$\frac{6}{5} = 1.2$	$\frac{4}{3} = 1.\overline{3}$	$\frac{13}{12} = 1.08\overline{3}$
10	$\frac{38}{27} = 1.\overline{407}$	$\frac{66}{53} \approx 1.245283$	$\frac{38}{27} = 1.\overline{407}$	$\frac{23}{20} = 1.15$
11	$\frac{22}{15} = 1.4\overline{6}$	1.290735	$\frac{22}{15} = 1.4\overline{6}$	$\frac{43}{36} = 1.19\overline{4}$
12	$\frac{14}{9} = 1.\overline{5}$	$\frac{4}{3} = 1.\overline{3}$	1.553571	$\frac{59}{48} = 1.2291\overline{6}$
13	$\frac{33}{20} = 1.65$	$\in [1.3620, 1.4211]$	$\frac{33}{20} = 1.65$	≈ 1.258772
14	$\frac{111}{64} = 1.734375$		$\frac{111}{64} = 1.734375$	≈ 1.298361

TABLE 1. The maximum critical threshold value for complete simple games restricted to strong or proper games.

Obviously we have the inequalities $c_{\mathcal{C}}^{ps}(n) \leq c_{\mathcal{C}}^p(n) \leq c_{\mathcal{C}}(n)$ and $c_{\mathcal{C}}^{ps}(n) \leq c_{\mathcal{C}}^s(n) \leq c_{\mathcal{C}}(n)$. Since adding an additional player to an arbitrary complete simple game, which is winning on its own, yields a strong complete simple game with equal critical threshold value, we also have $c_{\mathcal{C}}^s(n) \geq c_{\mathcal{C}}(n-1)$, i.e. Conjecture 2 would imply $c_{\mathcal{C}}^s(n) \in \Theta(\sqrt{n})$. Looking at the numerical values of Table 1 one might conjecture $c_{\mathcal{C}}^p(n) \leq c_{\mathcal{C}}^s(n)$ for all n. It would be very nice to have a good lower bound construction for $c_{\mathcal{C}}^{ps}(n)$, which then would imply lower bounds for $c_{\mathcal{C}}^p(n)$, $c_{\mathcal{S}}^{ps}(n)$, and $c_{\mathcal{S}}^p(n)$.

Lemma 15. *For all $k \geq 2$ we have $c_{\mathcal{S}}^s(2k) \geq \frac{k}{2}$.*

Proof. Consider the k coalitions $S_i := \{2i-1, 2i\}$ for $1 \leq i \leq k$ and the 2^k coalitions $\{a_1, \ldots, a_k\}$ with $a_i \in S_i$. Let us denote the latter set of coalitions by $\mathcal{A}$. We can easily check that those coalitions form an antichain so that we can arbitrarily prescribe for each coalition whether it is winning or losing and there exist at least one simple

$\mathbf{n}$	$\mathbf{c_S(n)}$	$\mathbf{c_S^p(n)}$	$\mathbf{c_S^s(n)}$	$\mathbf{c_S^{ps}(n)}$
5	$\frac{6}{5} = 1.2$	1	1	1
6	$\frac{3}{2} = 1.5$	$\frac{4}{3} = 1.\overline{3}$	$\frac{3}{2} = 1.5$	1
7	$\frac{12}{7} \approx 1.714286$	$\frac{7}{5} = 1.4$	$\frac{5}{3} = 1.\overline{6}$	$\frac{4}{3} = 1.\overline{3}$
8	2	$\frac{3}{2} = 1.5$	2	$\frac{7}{5} = 1.4$
9	$\frac{20}{9} = 2.\overline{2}$	$\frac{5}{3} = 1.\overline{6}$	$\frac{11}{5} = 2.2$	$\frac{3}{2} = 1.5$

TABLE 2. The maximum critical threshold value for simple games restricted to strong or proper games.

game χ meeting those conditions. Here we require that the coalitions S_i are winning the coalitions $\{a_1, \ldots, a_k\}$ are winning if and only if $a_1 = 1$, $a_2 = 3$ or $a_1 = 2$, $a_2 = 4$. Since the coalitions S_i are winning we have

$$k \leq \sum_{i=1}^{k} w(S_i) = \sum_{i=1}^{n} w_i.$$

Since the coalitions in $\mathcal{A} \cap L$, where L denotes the set of losing coalitions, contain each voter with equal frequency, we have

$$2^{k-1}\alpha \sum_{A \in \mathcal{A} \cap L} w(A) = \left(\sum_{i=1}^{n} \right) \cdot \frac{2^k}{2} \cdot \frac{k}{2k}.$$

Combining both inequalities gives $\alpha \geq \frac{k}{2}$. □

We remark that we have $c_S^s(n) \leq c_S(n)$ so that the bound from Lemma 15 is tight if Conjecture 1 is true.

Lemma 16. *For all $k \geq 2$ we have $c_S^s(2k+5) \geq 1 + \frac{k(k+1)}{2k+1}$.*

Proof. We will construct a class of examples by prescribing for some coalitions whether they are winning or losing. For $1 \leq i \leq 2k$ we require that the coalitions $\{i, i+1\}$ are winning. Let $B := \{2i - 1 \mid 1 \leq i \leq k + 1\}$ and $R := \{2i \mid 1 \leq i \leq k\}$. Next we require

$$\begin{aligned}
\{2k+2, 2k+4\} \cup B \in L & \qquad \{2k+3, 2k+5\} \cup R \in W, \\
\{2k+3, 2k+5\} \cup B \in L & \qquad \{2k+2, 2k+4\} \cup R \in W, \\
\{2k+2, 2k+5\} \cup B \in W & \qquad \{2k+3, 2k+4\} \cup R \in L, \\
\{2k+3, 2k+4\} \cup B \in W & \qquad \{2k+2, 2k+5\} \cup R \in L,
\end{aligned}$$

where L denotes the set of losing coalitions and W denotes the set of losing coalitions. The linear program for the computation of the critical threshold value restricted on the mentioned coalitions has an optimal solution of $1 + \frac{k(k+1)}{2k+1}$. □

We remark that the lower bound from Lemma 16 misses the value from Conjecture 1 only by $\frac{1}{n(n-4)}$. Since the computed exact values for $c_S(n)$ from Table 2 coincide with the lower bounds from Lemma 15 and Lemma 16, we conjecture that they are tight.

Unfortunately we can not use duality to obtain upper bounds for proper simple games from those for strong simple games. To this end let us consider the class of examples from the proof of Lemma 15. We observe that all coalitions of cardinality at least $k+1$ are winning so that each winning coalition of the dual game, which is strong, has a cardinality of at least k. Thus we may choose weights $w_i = \frac{1}{k}$ for all voters so that the weight of each losing coalition is at most 2 while the original game has a critical threshold value of $\max(1, \frac{k}{2})$.

Lemma 17. *For $n \geq 3$ we have $c_{\mathcal{B}}^p(n) = n$.*

Proof. Of course we have $c_{\mathcal{B}}^p(n) \leq c_{\mathcal{B}}(n) = n$. A proper example achieving this bound is given by the Boolean game whose winning coalitions coincide with the coalitions of size one. □

Lemma 18. *We have $c_{\mathcal{B}}^s(n) = \max(1, n-1)$ for all $n \in \mathbb{N}$.*

Proof. Since the empty set is a losing coalition, its complement, the grand coalition, has to be winning. Thus every losing coalition consists of at most $n-1$ members. Choosing weights $w_i = 1$ for all voters gives a feasible weighting with $\alpha \leq n-1$. For the other direction consider the strong *game* in $\mathcal{B}_n$ with $n \geq 3$, whose losing

n	$c_{\mathcal{B}}(n)$	$c_{\mathcal{B}}^{p}(n)$	$c_{\mathcal{B}}^{s}(n)$	$c_{\mathcal{B}}^{ps}(n)$
1	1	1	1	1
2	2	1	1	1
3	3	3	2	2
4	4	4	3	3
5	5	5	4	4
6	6	6	5	5
7	7	7	6	6
8	8	8	7	7
9	9	9	8	8

TABLE 3. The maximum critical threshold value for Boolean games restricted to strong or proper games.

coalitions are the empty set and the coalitions of size $n-1$. Since all coalitions of size 1 are winning, the weights of the players have to be at least one so that the losing coalitions of cardinality $n-1$ have a weight of at least $n-1$. □

Lemma 19. *We have $c_{\mathcal{B}}^{ps}(n) = \max(1, n-1)$ for all $n \in \mathbb{N}$.*

Proof. Since $c_{\mathcal{B}}^{ps}(n) \leq c_{\mathcal{B}}^{s}(n) = \max(1, n-1)$ it suffice to construct an example whose critical threshold value reaches the upper bound. To this end we define the strong and proper Boolean game χ for $n \geq 3$ as follows: The empty coalition is loosing, the grand coalition is winning, coalitions with sizes between one and $\frac{n-1}{2}$, coalitions with sizes between $\frac{n+1}{2}$ and $n-1$ are winning, and coalitions of cardinality $\frac{n}{2}$ are winning if and only if they contain voter 1. □

A.2. **Restrictions on the number of shift-minimal winning vectors.** Using the described ILP approach we may also exactly determine the maximal alpha-values $c_{\mathcal{C}}(n, 1)$ of complete simple games with n players and a single shift-minimal winning coalition. As all complete simple games with at most six voters are roughly weighted we have $\tilde{s}(n, 1) = 1$ for $n = 6$. The next exact values are given by

- $c_{\mathcal{C}}(7, 1) = \frac{10}{9} \approx 1.111111$: (2, 5); (1, 2); (2, 0), (0, 5)
- $c_{\mathcal{C}}(8, 1) = \frac{6}{5} = 1.2$: (2, 6); (1, 2); (2, 0), (0, 6)
- $c_{\mathcal{C}}(9, 1) = \frac{15}{11} \approx 1.272727$: (2, 7); (1, 2); (2, 0), (0, 7)
- $c_{\mathcal{C}}(10, 1) = \frac{4}{3} \approx 1.333333$: (2, 8); (1, 2); (2, 0), (0, 8)
- $c_{\mathcal{C}}(11, 1) \approx 1.41176470588$: (3, 8); (1, 3); (3, 0), (0, 8)
- $c_{\mathcal{C}}(12, 1) = \frac{3}{2} = 1.5$: (3, 9); (1, 3); (3, 0), (0, 9)
- $c_{\mathcal{C}}(13, 1) \approx 1.57894736842$: (3, 10); (1, 3); (3, 0), (0, 10)
- $c_{\mathcal{C}}(14, 1) = \frac{33}{20} = 1.65$: (3, 11); (1, 3); (3, 0), (0, 11)

Here we also state the cardinality vector, the list of shift-minimal winning vectors, and the list of shift-maximal losing vectors of an example reaching the upper bound $c_{\mathcal{C}}(n, 1)$, respectively.

We can enhance our ILP formulations to additionally treat conditions on the shift-minimal winning coalitions easily. For $S \subseteq N$ we introduce a binary variable s_S with the meaning that $s_S = 1$ iff coalition S is a shift minimal winning coalition. As conditions we have

$$s_S \leq x_S$$
$$s_S \leq 1 - x_{S'} \quad \forall S' \prec S : \nexists S'' : S' \prec S'' \prec S$$
$$-x_S + \sum_{S' \prec S : \nexists S'' : S' \prec S'' \prec S} x_{S'} + s_S \geq 0.$$

By setting

$$\sum_{S \subseteq N} s_S = r$$

we can easily formulate exact values, lower or upper bounds for the number r of shift-minimal winning coalitions. To be able to express the number t of equivalence classes of voters we introduce the functions $\varphi_i : 2^N \to \{0, 1\}$

for all $1 \leq i \leq n-1$ where $\varphi_i(S) = 1$ iff a shift-minimal winning coalition S implies that voter i and voter $i+1$ have to be in different equivalence classes. We use binary variables p_i for $1 \leq i \leq n-1$ and the constraints

$$p_i \geq s_S \cdot \varphi_i(S) \quad \forall S \subseteq N, 1 \leq i \leq n-1$$
$$p_i \leq \sum_{S \subseteq N} s_S \cdot \varphi_i(S) \quad \forall 1 \leq i \leq n-1$$
$$\sum_{i=1}^{n-1} p_i = t.$$

Let us denote by $c_{\mathcal{C}}(n,r,t)$ the maximum critical α-value of a complete simple game with r shift-minimal winning coalitions consisting of n voters being partitioned into t equivalence classes. We have $c_{\mathcal{C}}(7,1,2) = \frac{10}{9}$, $c_{\mathcal{C}}(7,2,2) = \frac{17}{15}$, $c_{\mathcal{C}}(7,3,2) = \frac{8}{7}$, $c_{\mathcal{C}}(7,4,2) = \frac{15}{15}$, $c_{\mathcal{C}}(7,5,2) = \frac{19}{17}$, and there are no such games for $r \geq 6$. Examples of the corresponding sets of the shift-minimal winning coalitions are given by $\{35\}$, $\{41, 70\}$, $\{44, 49, 67\}$, $\{43, 44, 49, 67\}$, and $\{31, 60, 86, 88, 96\}$, respectively.

APPENDIX B. COMPARISION OF DIFFERENT ILP SOLVERS

We give some running time information for different ILP solvers in Table 4.

	CPLEX		CPLEX*		Gurobi 4.0.0		Gurobi 4.5.0	
n	nodes	seconds	nodes	seconds	nodes	seconds	nodes	seconds
7	459	0.4	1113	0.7	975	0.5	582	0.4
8	3721	10	2271	3.7	1900	1.7	1715	1.8
9	3594	25	3297	14	3153	15	3724	12
10	11799	154	8974	94	12008	123	20988	83
11	33312	2052	42340	2131	29049	349	102306	979
12	55180	*32379*			45752	1301	215336	5403
13	94982	*304255*			64962	4318	83393	20408
14					97532	22230		
15					152047	134118		
16					308240	230964		

TABLE 4. Comparing different ILP solvers (using 4 available kernels).

The solvers CPLEX 12.1.0 and Gurobi 4.0.0 are used with the standard parameter settings. Using the tuning option of CPLEX we find out that the parameter settings `mip strategy heuristicfeq -1`, `mip strategy probe -1`, and `mip strategy variableselect 4` might be better suited. The results are summarized under column CPLEX* of Table 4. We may say that these parameter settings might be good for small instances but can not be generalized to larger instances easily.

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